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A Survey on Macroscopic Quantum Phenomena in Superconductive Devices

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Abstract

This work originated from a summer spent studying some rudiments of the theory of superconductivity and some of its applications, as a summer student at Fermi National Accelerator Laboratory in Batavia, Illinois. This document is a report for my supervisor (Dr. Frank Nezrick) of my month and a half of work and I hope that it might be useful to some other student, at least as a list of a small part of the existing bibliography on the subject.

On the bibliography a comment should be made. Sometimes the original papers have been cited, but if I haven't gone through them they have been put in a footnote, instead of being listed in the bibliography at the end of the document.

The document starts with a basic review of Quantum Mechanics (Chapter 1), where the results most often needed in the rest of the document have been stated. The main subject starts with Chapter 2, where the qualitative features of the theory of superconductivity (the so-called BCS theory) are analyzed and some of the basic equations that describe the characteristic phenomena of this theory are derived. In Chapter 3, the tunnel junctions play the major role, a semiconductor analog following [15] has been used, because it appears to me that the physics which is behind tunnelling comes out in a very natural way, without requiring me to learn all the mathematical apparatus which is behind the mani-body theory of tunnelling. In Chapter 4 I have described the Josephson junction, the Cooper pairs' tunnelling mechanism and I have tried to sketch a few applications; unfortunately I didn't have much time left to get deep into the devices. Some details or more mathematical topics which I, nonetheless, found interesting have been relegated to the Appendices.

I would like to thank Dr. Frank Nezrick (Physics Section) for the opportunity he gave me to learn all these subjects, all new to me, and for his guidance and suggestions during this work. I am very grateful to Dr. Drasko Jovanovic (Physics Section) and Dr. Chris Hill (Theoretical Physics) for the time they spent talking with me. Finally I would like to thank Lee Robbins of the Main Library staff, without whom it would have been almost impossible to obtain some of the bibliography.

Contents

Chapter 1

Quantum mechanics: a brief review

1.1 The time-dependent Schrödinger equation

In this chapter I would like to review, without even attempting to give any proof, the results of quantum mechanics that I will need in the rest of the paper. The reader who wants more details on some particular subject, should refer to his favourite quantum mechanics book.

The equation of motion for a quantum mechanical system is the *Schrödinger equation*, which states:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right)\Psi(\mathbf{r},t) = \hat{\mathcal{H}}\Psi(\mathbf{r},t), \tag{1.1}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian operator [1, Chapter 2]. To setup the Schrödinger equation for most systems it is enough to:

- Write the classical Hamiltonian $\mathcal{H}(q_1,\ldots,q_R;p_1,\ldots,p_R;t)$ thus finding the classical equation
$$E = \mathcal{H}(q_1,\ldots,q_R;p_1,\ldots,p_R;t). \tag{1.2}$$
- Perform the substitutions
$$E \to i\hbar\frac{\partial}{\partial t} \qquad p_r \to \frac{\hbar}{i}\frac{\partial}{\partial q_r} \qquad (r=1,2,\ldots,R), \tag{1.3}$$
on both sides of equation (1.2) and write down that these two quantities, considered as operators, give the same result when acting on Ψ.

The equation thus obtained is the *Schrödinger equation* for the quantum system [1, §II-15]:

$$i\hbar\frac{\partial}{\partial t}\Psi(q_1,\ldots,q_R;t) = \hat{\mathcal{H}}\left(q_1,\ldots,q_R;\frac{\hbar}{i}\frac{\partial}{\partial q_1},\ldots,\frac{\hbar}{i}\frac{\partial}{\partial q_R};t\right)\Psi(q_1,\ldots,q_R;t). \quad (1.4)$$

Let's consider, for example the case of a particle in an electro-magnetic field.The Lagrangian can be derived from a velocity-dependent potential and is found to be [2, §1-5]

$$L = T - q\phi + q\mathbf{A}\cdot\mathbf{v}, \quad (1.5)$$

where T is the kinetic energy. From (1.5) it is an easy step to write the Hamiltonian [2, page 222]

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi, \quad (1.6)$$

where $\mathbf{p}$ is the canonical momentum. Using the "rules" given above, the Schrödinger equation for the particle reads:

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\phi\right]\Psi. \quad (1.7)$$

For more details and further examples see [1, §II-14].

1.2 The time-independent Schrödinger equation

Let's now assume that the Hamiltonian $\mathcal{H}$ does not depend on time. This is the case of a conservative system, i.e. a system whose total energy E is a constant of the motion. We look for a solution Ψ of the Schrödinger equation representing a state with well defined energy $E = \hbar\omega$, so we look for solutions of the form:

$$\Psi = \psi e^{-i\frac{Et}{\hbar}}, \quad (1.8)$$

where ψ depends upon the configuration coordinates but not upon the time. Substitution of this expression in the (1.4) leads to:

$$\hat{\mathcal{H}}\psi = E\psi, \quad (1.9)$$

which is the *time-independent Schrödinger equation.* We will call ψ the wave function or the wave function of the stationary state, although the real wave function differs from ψ for the phase factor $\exp(-iEt/\hbar)$[1, pages 71–74].

1.3 The equation of continuity for probabilities

An important part of the Schrödinger equation for a single particle is the idea that the probability of finding the particle somewhere is the absolute square of the wave function (see for example [1, Chapter IV]). It is also characteristic of quantum mechanics that probability is conserved locally [3, §21-2], which means that if the likelihood of finding the particle somewhere increases while it decreases somewhere else, then something in between must be happening. This is usually expressed in a more mathematical way stating that if the probability in a volume is changing, there must be a "flow" of something across the boundary of the volume. This "something" is usually called a *current* even if it has nothing to do with the electric current. We can thus write an *equation of continuity* for the probability,

$$\nabla \cdot \mathbf{J} + \frac{\partial P}{\partial t} = 0, \tag{1.10}$$

where P is just:

$$P(\mathbf{r}, t) = \Psi^*(\mathbf{r}, t)\Psi(\mathbf{r}, t). \tag{1.11}$$

It is not obvious from what we have said that such a current exists or what is its analytical expression, but I am not going to discuss this topic any further here; one way to do it can be found in [3, §21-2]. I have worked out another way to find $\mathbf{J}$ using the invariance properties of the Lagrangian density for the Schrödinger equation (Noether's theorem), analogously to what is done in electro-magnetism to show that the freedom in the choice of the gauge for the vector and scalar potential implies charge conservation, and you can read this in Appendix A.

Chapter 2

The macroscopic wave function

2.1 The superconducting state

The electrical resistivity of many metals, alloys and some ceramics drops suddenly to zero if the specimen of the material is cooled to a sufficiently low temperature. This property was first observed by Kammerlingh Onnes in Leiden in 1911. This is not the only astonishing phenomenon occuring in superconductors, they have anomalous magnetic behaviour too. If placed in a weak magnetic field, a bulk superconductor will behave as a perfect diamagnet, shielding completely its interior from the magnetic induction. When a specimen is placed in a magnetic field and is then cooled through the transition temperature for superconductivity, the flux originally present is expelled from the specimen, this is called the *Meissner effect.*

The temperature interval in which the resistivity change occurs is very narrow. We say that at a critical temperature T_c the sample undergoes a phase transition from the normal to the superconducting state. There have been many attempts to place all of the above phenomenology in a theoretical framework but until 1957 the basic mechanisms that "build" superconductivity were not completely understood. In that year, in two papers[1] J. Bardeen, L. N. Cooper and J. R. Schrieffer published what is now called *the BCS theory of superconductivity.* These papers explained the observed phenomenology without making any macroscopic assumption, i.e. building the theory from the microscopic interaction of the electrons in the lattice.

2.2 The meaning of the wave function

The statistical meaning of the wavefunction is well-known, but I would like to remind that the wave function ψ does not describe a smeared out particle. The particle is

[1] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957)

either here or somewhere else, with different probabilities, but wherever it is, *the whole particle is there*. Thus an interpretation of the wave function as a charge density, for instance, in the case of a charged particle is completely out of question.

Things change drastically when we consider a situation in which there are an enormous number of particles *in exactly the same state*, the probability of finding any one of them at a given place is $\psi^*\psi$. But due to the big number of particles, if I look in a volume element $dx\,dy\,dz$ I will usually find a number of particles given by $\mathcal{N}\psi^*\psi\,dx\,dy\,dz$, where $\mathcal{N}$ is the total number of particles in the system. So in this situation, $\mathcal{N}\psi^*\psi$ can indeed be interpreted as the density of particles [3, §21-4].

If all the particle constituting the system carry the same charge, then we can go a step further and use the product $\mathcal{N}\psi^*\psi$ to find the charge density. We know that the charge density is

$$\rho = q\mathcal{N}\psi^*\psi, \tag{2.1}$$

if q is the charge of a single particle; unfortunately people don't like writing too much, and so in the literature you will find the product $\psi^*\psi$ referred to as "the charge density". We will adopt this notation too, from now on, writing the wave function as:

$$\tilde{\psi}(\mathbf{r},t) \stackrel{\text{def}}{=} \sqrt{\rho(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}, \tag{2.2}$$

(I decided to put a tilde over ψ to avoid any confusion between the wave function and this "modified wave function") where ρ and θ are functions of real variables, so that $\tilde{\psi}^*\tilde{\psi} = \rho$ (any $\tilde{\psi}$ can be written in this way, it is just like going from cartesian coordinates to polar coordinates on the plane). Under these assumptions the equation of continuity for probabilities becomes the familiar equation of charge conservation in electrodynamics

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \tag{2.3}$$

where $\mathbf{J}$ is now the electric current density and $\rho(\mathbf{r},t) = \tilde{\psi}^*(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)$ is the electric charge density. We have thus established a very tight relationship between the probability current and the electric current density on the sole basis that "the same equations have the same solutions".

The role of ρ in the wave function is clear, what about θ ? Let's see what happens when we substitute the equation (2.2) into the expression (A.11) for the electric current density and express the result in terms of the new variables ρ and θ [3, §21-5]:

$$\mathbf{J} = \frac{1}{2m}\left[\tilde{\psi}^*\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\tilde{\psi} + \tilde{\psi}\left(-\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\tilde{\psi}^*\right], \tag{2.4}$$

notice that $\mathbf{J}$ is indeed a real quantity being the average between a complex expression and its complex conjugate. The explicit calculation is done in Appendix B, and the

result is:

$$\mathbf{J} = \frac{\hbar}{m}\left(\nabla\theta - \frac{q}{\hbar}\mathbf{A}\right)\rho. \tag{2.5}$$

This equation is independent from Maxwell equations and it will turn out that this is all we need to explain flux quantization and the Meissner effect in a superconductor even if, at this stage, it is not yet clear why it should describe the superconducting behaviour.

Incidentally, the equation for the current can be analyzed a little nicer, when you think that the current density $\mathbf{J}$ is in fact the charge density times the velocity of the fluid of electrons. In fact this means that

$$m\mathbf{v} = \hbar\nabla\theta - q\mathbf{A}, \tag{2.6}$$

which means that $\hbar\nabla\theta$ is just the canonical momentum in the electro-magnetic field.

2.3 Brief insights in superconductivity

2.3.1 The BCS theory: qualitative features

Since now we have been developing the quantum mechanics of a system made of a big number of particles all described by the same wave function, but a theory of superconductivity should be based on the behaviour of electrons in the lattice and we know that electrons, being fermions, obey the Pauli exclusion principle, so they cannot be described by a macroscopic wave function. It seemed that there was no way of applying all the nice features we have seen until, with the pioneering works of H. Frölich first, who introduced the idea of electron-phonon-electron interaction as the source of superconductivity, and then L. Cooper, who advanced an idea on how a pairing mechanism could take place in the lattice provided there was an effective attractive potential (even very weak) so that bound pairs could arise at a sufficiently low temperature thus having the formation of units, later called "Cooper pairs", of 0 total spin and 0 total center of mass momentum, the final picture of superconductivity begun forming. The interaction roughly works like this: one electron interacts with the lattice and deforms it (creation of a phonon); a second electron sees the deformed lattice and adjusts itself to take advantage of the deformation to lower its energy (absorption of a phonon) [4, page 357]. Thus the second electron interacts with the first one indirectly via the phonon field. This mechanism is characterized by a very small binding energy so that very low temperature have to be reached in order to see its effects.

The pairs are not held together as a point particle, in fact the average distance between the two electrons forming the pair is some orders of magnitude bigger than

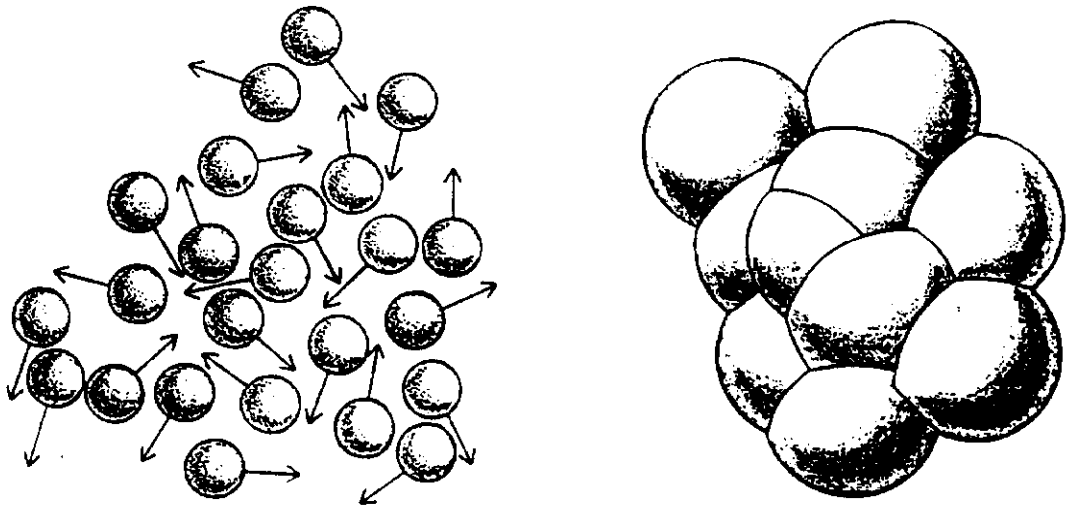

Figure 2.1: Many-Cooper pair problem: (a) incorrect and (b) correct picture.

the average separation between the pairs. Thus within the region occupied by a single pair, will be found the centers of many (millions, or more) pairs [5, pages 739–743]. This last argument suggests that Cooper pairs cannot be treated as independent particles. they are spatially interlocked in a very intricate manner, which is essential to the stability of the state (Figure 2.1).

The Schrödinger equation for a pair will be very similar to the one for a single charged particle in an electro-magnetic field:

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - q^*\mathbf{A}\right)^2 + q^*\phi\right]\psi, \tag{2.7}$$

except from the fact that we will have to substitute q with $q^* = -2e$ for the charge of the pair (e is the modulus of the charge of the electron), and m with m^* which is an effective mass which takes care of the non-independence of Cooper pairs (just as it is done with semiconductors, m^* is the mass of the "dressed" pair, i.e. it accounts for the interaction with the lattice and with the other pairs via a screened Coulomb interaction). We will not use directly this equation, but we will make a very large use of the (2.5), modified for Cooper pairs, which comes directly from the (2.7). Another important point is that we will not consider the effects of non-bound electrons, i.e. we will do every calculation in the zero temperature approximation.

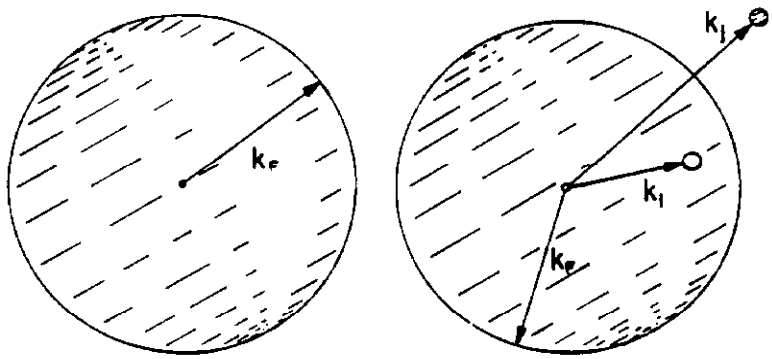

Figure 2.2: An excitation of the normal system.

2.3.2 The BCS ground state and the energy gap

In a metal, in the free electron approximation, the ground state (state of lowest energy) is just the filled Fermi sea [4, Chapter 7] this can be pictured in momentum space as the filling of a Fermi sphere. In the ground-state wave function there is no correlation between electrons with opposite spin and only a statistical correlation—due to the Pauli exclusion principle—between electrons with the same spin. Single particle excitations are given by wave functions identical to the ground state except that one electron states with $k_i < k_F$ are replaced with states $k_j > k_F$, leaving a vacancy below the Fermi surface and placing excited electrons above, as in Figure 2.2. The energy difference between the ground state and the excited states can be made arbitrarily small as there are many accessible states right outside the Fermi surface [6].

Coming to the superconducting ground-state, the recognition of the basic electron interaction mechanism doesn't remove the major difficulty of the theory, namely that the correlation energy is so very much smaller than almost any other contribution to the total ground-state energy. A very drastic assumption has to be made, all interactions except the crucial one are the same for the normal and superconducting phase at 0 K. Thus, taking as the zero energy the ground-state for the normal phase, the BCS theory proceeds to calculate the superconducting ground-state energy as being due only to the correlation between Cooper pairs and the screened Coulomb interaction [7, §11.4]. Considering only the electrons in a narrow shell of $\hbar\omega_q$ thickness (ω_q is the frequency of a phonon that carries a momentum q, so $\hbar\omega_q$ is the average energy of the lattice) around the Fermi energy as responsible for the pairing interaction, the effective potential can be written (measuring electron energies from the Fermi surface,

and calling ϵ_k the energy of an electron in the state k)

$$V_{kk'} \stackrel{\text{def}}{=} \begin{cases} V & \text{for } |\epsilon_k|,\ |\epsilon_{k'}| \leq \hbar\omega_q \\ 0 & \text{elsewhere.} \end{cases} \tag{2.8}$$

The basic similarity of the superconducting characteristics of widely different metals implies that the responsible interaction cannot crucially depend on details of individual substances. In the BCS theory is therefore made the simplifying assumption that $V_{kk'}$ is isotropic and constant for all the electrons in the narrow shell. This simplification leads to virtually identical predictions for the magnitudes of all characteristic quantities in terms of reduced coordinates. Any empirical deviation from such complete similarity is, therefore, an indication of this oversimplification.

The fundamental qualitative difference between the superconducting and normal ground-state wave function is produced when the large degeneracy of the single particle electron levels in the normal state is removed [6, pages 281–283] lowering the ground-state of degeneracy one by a finite amount under the excited states, thus making it very stable, creating a finite energy gap in the excitation spectrum (see Appendix C for the details of the mechanism).

The energy gap is the most fundamental concept to understand the dynamics of superconductivity; it ensures the stability of persistent currents, the Meissner effect and it is crucial in tunnelling phenomena.

Let's see now, for example, how the energy gap can account for the stability of the supercurrents, i.e. the currents made of Cooper pairs. Consider a crystal lattice of mass M which contains an imperfection, such as a phonon or an impurity. Let the lattice flow vith velocity $\mathbf{v}$ relative to the electron gas. The relative velocity will decrease if excitations in the electron gas can be generated by collisions. If there is such a collision creating an excitation of energy $\lambda_{\mathbf{k}}$ and momentum $\hbar\mathbf{k}$, we must have the conservation of energy and momentum so,

$$\frac{1}{2}Mv^2 = \frac{1}{2}M(v')^2 + \lambda_{\mathbf{k}}, \tag{2.9}$$

$$M\mathbf{v} = M\mathbf{v}' + \hbar\mathbf{k}. \tag{2.10}$$

If we combine these two equations we have

$$0 = \hbar\mathbf{k}\cdot\mathbf{v} + \frac{\hbar^2k^2}{2M} + \lambda_{\mathbf{k}}. \tag{2.11}$$

If we now let $M \to \infty$ we can neglect the second term in the right hand side. Thus the lowest value of v for which the equation $\lambda_{\mathbf{k}} = \hbar\mathbf{k}\cdot(-\mathbf{v})$ can be satisfied is

$$v_c = \min\left(\frac{\lambda_{\mathbf{k}}}{\hbar k}\right). \tag{2.12}$$

If there is an energy gap, $\lambda_{\mathbf{k}} > 0$ even for the lowest excitation, so also $v_c > 0$. Thus superconducting currents can flow with velocities lower than v_c without the risk of dissipation of energy by excitation of electrons from Cooper pairs via interaction with the lattice. This is also a partial explanation for the very poor heat conductivity in superconductors.

We are now facing a new phenomenon, usually we have been thinking of an energy gap as *preventing* currents from flowing (if the valence and conduction bands of a material are separated by a finite gap whose thickness is much greater than kT, you are dealing with an insulator), but now we are told that a superconductor is a perfect conductor *because it has an energy gap!* What's happening ? There is a very important difference between the energy gap of an insulator, and the one in a superconductor, it is about the origin of the gap. In the insulator, the gap is tied to the lattice, it comes from the periodic disposition of positive ions in the crystal lattice so it is an essentially *static* gap, the electrons don't affect it in any important way with all their wiggling around. In the superconductor, instead, the energy gap springs from the interactions that have been turned on inside the Fermi gas; it is thus an extremely *dynamical* parameter tied to the electrons.

It is shown in [8, §10.9] that when a net current is flowing through a superconductor, which means that the Cooper pairs instead of having total momentum 0 ($\mathbf{k}_1 = \mathbf{k}$, $\mathbf{k}_2 = -\mathbf{k}$), have total momentum $\hbar\mathbf{Q}$ ($\mathbf{k}_1 = \mathbf{k}+\mathbf{Q}$, $\mathbf{k}_2 = -\mathbf{k}+\mathbf{Q}$) (see Appendix C for more details on Cooper pairs), the energy gap shifts up as a whole by $2[\hbar^2 Q^2/(2m_e)]$ (where m_e is the mass of the electron). This is the reason why we have described the flow of the electron gas in the lattice in the (2.9) and (2.10), as seen by an observer tied to the electrons, instead of choosing the reference frame of the lattice: the energy gap looks always the same when it is viewed from the electrons' reference frame.

2.3.3 The Meissner effect

Now we can start describing some of the phenomena of superconductivity. We have already given an explanation for the zero resistance behaviour in terms of the energy gap; we want to turn to magnetic properties now.

It is clear that if supercurrents can flow without any dissipation, magnetic fields will not penetrate the material in the superconducting state. If, as you start the magnetic field, any of it built up in the superconductor, there would be a rate of change of the magnetic flux which would create an electric field which, by Lenz's law would oppose the field change. Since electrons can move without any resistance, an arbitrarily small electric field would produce just the right current to cancel exactly the magnetic flux change. So if you turn on the field *after* the sample has been cooled to the superconducting state, it will exclude the field.

A more interesting phenomenon appears if you cool down the sample *in* a magnetic

field: as soon as the sample becomes superconducting *the field is expelled*, in other words it starts a current in just the right amount to screen itself completely from the external magnetic field. This time there is no rate of change of the magnetic flux, so classical electro-magnetism fails in giving an explanation for what is happening inside the superconductor [3, §21-6].

We can see the reason for this in the equations that we have already written down in a simple case. Let's consider a long cylinder of superconductor in stationary conditions; we can write the equation of charge conservation as $\nabla \cdot \mathbf{J}_s = 0$, where $\mathbf{J}_s$ is the superconducting current density, with the boundary condition $\mathbf{J}_s \cdot \hat{\mathbf{n}} = 0$ on the surface, where $\hat{\mathbf{n}}$ is the direction normal to the boundary and pointing towards the outside,which means that surface currents don't "escape" from the superconductor. Now taking the divergence of equation (2.5) we find, supposing that ρ is constant throughout the superconductor (which is quite reasonable in ordinary conditions, otherwise there would be a terrific repulsion pushing the electrons apart):

$$\nabla^2\theta - \frac{q^*}{\hbar}\nabla \cdot \mathbf{A} = 0, \tag{2.13}$$

and choosing the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$ we finally have the equation for θ

$$\nabla^2\theta = 0. \tag{2.14}$$

Let's turn to the boundary conditions now, in our simple geometry let's suppose to have a magnetic field parallel to the cylinder, the vector potential will then be as in Figure 2.2. It is thus evident that $\mathbf{A} \cdot \hat{\mathbf{n}} = 0$ and so we finally have the boundary value problem with Neumann conditions:

$$\begin{cases} \nabla^2\theta &= 0 \\ (\nabla\theta) \cdot \hat{\mathbf{n}} &= 0 \quad \text{at the surface.} \end{cases} \tag{2.15}$$

The solution for (2.15) is $\theta = \text{const.}$ and so equation (2.5) becomes

$$\mathbf{J}_s = -\frac{q^*\rho}{m^*}\mathbf{A}. \tag{2.16}$$

This equation is called the *London equation* and was originally proposed by H. London and F. London to explain the experimental observations, long before the quantum mechanical theory came forth.

There is another interesting way to find the London equation, considering the quantum mechanical electric current [8, pages 268–269]

$$\begin{aligned} \mathbf{J}_s &= \frac{1}{2m^*}\left[\tilde{\psi}^*\left(\frac{\hbar}{i}\nabla - q^*\mathbf{A}\right)\tilde{\psi} + \tilde{\psi}\left(-\frac{\hbar}{i}\nabla - q^*\mathbf{A}\right)\tilde{\psi}^*\right] \\ &= \frac{\hbar}{2im^*}[\tilde{\psi}^*\nabla\tilde{\psi} - (\nabla\tilde{\psi})^*\tilde{\psi}] - \frac{q^*\rho}{m^*}\mathbf{A}, \end{aligned} \tag{2.17}$$

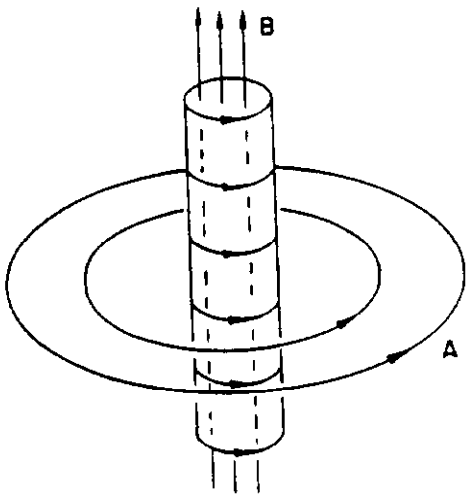

Figure 2.3: A long superconducting cylinder in an external magnetic field.

where I have set $\tilde{\psi}^*\tilde{\psi} = \rho$. If we have $\mathbf{A} = 0$ it is reasonable to assume $\mathbf{J}_s = 0$ and so we get

$$\mathbf{J}_s = \frac{\hbar}{2im^*}[\tilde{\psi_0}^*\nabla\tilde{\psi_0} - (\nabla\tilde{\psi_0})^*\tilde{\psi_0}] = 0. \tag{2.18}$$

When $\mathbf{A} \neq 0$, ψ_0 changes into ψ. If we suppose that $\psi \approx \psi_0$ then we find the London equation

$$\mathbf{J}_s = -\frac{q^*\rho}{m^*}\mathbf{A}. \tag{2.19}$$

We see then that if ψ is "rigid" in the sense that it doesn't change too much when a magnetic field is applied then we get London equation. But why should ψ be rigid ? From perturbation theory[2] we have

$$\psi = \psi_0 + \sum_{n\neq 0} \frac{\langle n|H_{\text{pert}}|0\rangle}{E_n - E_0}|n\rangle, \tag{2.20}$$

where E_0 is the ground-state energy and E_n is the energy of the excited state $|n\rangle$. If there is a gap between the ground-state energy and the first excited state then, in a first approximation, $E_n - E_0$ is large and $\psi \approx \psi_0$. We see that the energy gap can account for magnetic effects as well; this is just what happens in atoms,

[2]You can find a chapter on perturbation theory on almost any book on quantum mechanics: I have looked at it on [9].

where magnetic fields produce only higher orders corrections to the ground-state wave function (Zeeman effect) exactly for the same reason.

Let's turn back to the Meissner effect, we can now use the London equation in the equations of electromagnetism to find:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} = \mu_0 \frac{q^* \rho}{m^*} \mathbf{A} = \lambda^2 \mathbf{A}. \tag{2.21}$$

We can try to solve this equation and see what happens. In one dimension it has solutions $e^{\lambda x}$ and $e^{-\lambda x}$. These solutions mean that the vector potential has to decrease exponentially going inside the superconductor (we don't consider the solution $e^{\lambda x}$ because it diverges as $x \to \infty$). If the piece of metal is large compared to $\frac{1}{\lambda} \sim 0.1 \mu$m than the interior is completely free of field. We have thus explained the Meissner effect in this particularly simple geometry, for an analysis of the general case see Appendix D.

2.3.4 Flux quantization

If we consider now a ring or any multiply connected region instead of a lump, as we did before, whose thickness is much larger than $\frac{1}{\lambda}$ and see what happens if we start a magnetic field, cool down the metal to the superconducting state and turn off the magnetic field, even more dramatic effects take place. In the normal phase there will be a field in the body of the ring (Figure 2.4a), then as we cool it down the magnetic field is expelled from the material, but there will be some field in the hole (Figure 2.4b), and finally when we remove the magnetic field *the lines of field going through the hole are trapped* (Figure 2.4c). This is because the flux Φ inside the hole cannot decrease as [3, §21-7]

$$\frac{\partial \Phi}{\partial t} = -\oint \mathbf{E} \cdot d\mathbf{l} = 0. \tag{2.22}$$

So, as the field is removed a supercurrent starts flowing around the ring to keep the flux inside constant, but surprisingly enough the total flux trapped is quantized in integer multiples of

$$\Phi_0 \stackrel{\text{def}}{=} \frac{\pi \hbar}{e} \approx 2 \times 10^{-7} \text{gauss cm}^2, \tag{2.23}$$

where e is the electron charge. We can see why there is such a behaviour again starting from equation (2.5); well inside the body of the ring, the current density $\mathbf{J}_s$ is zero, so we find

$$\hbar \nabla \theta = q^* \mathbf{A}. \tag{2.24}$$

Let's now consider the line integral of both members of this equation around a closed loop which goes around the ring near the center of its cross-section, so that it is always

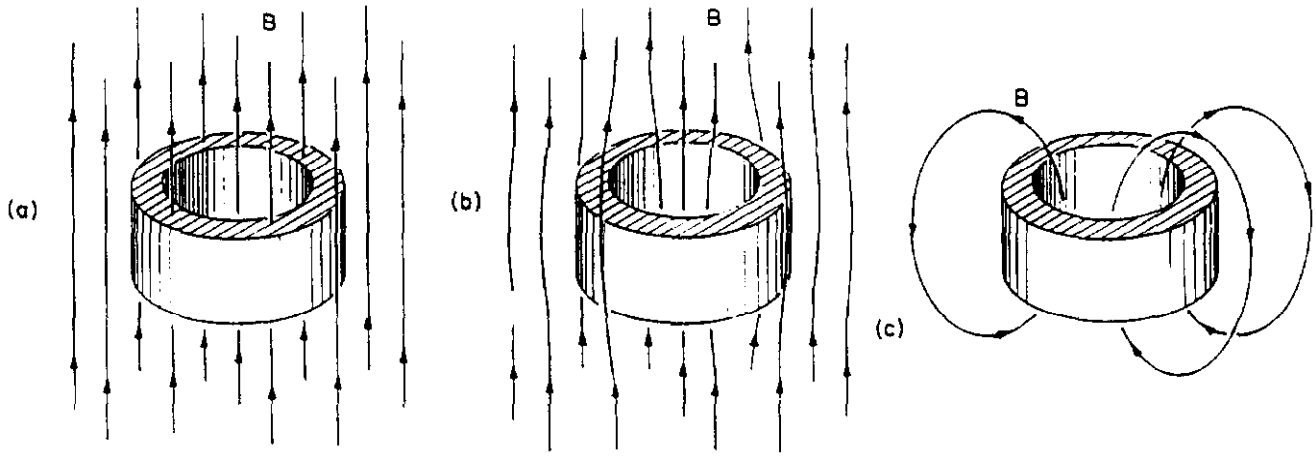

Figure 2.4: A ring in a magnetic field: (a) in the normal state; (b) in the superconducting state; (c) after the external field is removed.

distant enough from the surface

$$\hbar \oint \nabla\theta \cdot d\mathbf{l} = q^* \oint \mathbf{A} \cdot d\mathbf{l} = q^*\Phi, \tag{2.25}$$

where Φ is the flux of the magnetic induction $\mathbf{B}$ through the hole. The line integral *from one point to another is the difference of the values of the function at the two* points, but if we close the loop the line integral is not necessarily zero because the region is not simply connected, the only physical requirement we have is that $\tilde{\psi}$ has to be one valued, and this will happen if after one turn θ has changed by an integer multiple of 2π (remember that $\tilde{\psi} \stackrel{\text{def}}{=} \sqrt{\rho}e^{i\theta}$), so we have

$$2\pi n\hbar = q^*\Phi, \tag{2.26}$$

and thus:

$$\Phi = \frac{2\pi\hbar}{q^*}n = n\Phi_0, \tag{2.27}$$

which is just what we stated at the beginning. *The trapped flux must be an integer times* $\pi\hbar/e$. So what happens is that a supercurrent will start in the ring to bring the trapped flux to the nearest integer times Φ_0 (thus the supercurrent will either build or destroy some trapped flux according to its value).

Chapter 3

Tunnel junctions

3.1 Introduction: the semiconductor model

If the motion of a particle in the neighbourhood of a potential barrier is treated quantum mechanically, it is found that there is a finite probability that the particle can leak through the barrier even though its kinetic energy is less than the potential energy corresponding to the height of the barrier (for a simple introduction to the subject see [10, Chapter 4]). In other words a particle impinging on the barrier will not necessarily be reflected, but may pass through the barrier and continue its forward motion. Tunnelling effects were found to be very important in many fields [11, pages 34–38] already in the twenties and thirties, but superconductive tunnelling had to wait until 1960 to be discovered by Giaver[1] and Nicol et al.[2].

A tunnel junction connected by electrical leads to a tunable current source and a voltmeter is shown in Figure 3.1, the insulator width is usually around 50 Å or less. Junctions provide the most direct measurement of the energy gap by the observation of the tunnelling of electrons between a superconducting film and a normal one (or between two superconducting films) across an insulating barrier.

Considering tunnelling phenomena, it should be said that when a Cooper pair breaks into two "electrons", they still aren't free particles, they form single particle excitations of the BCS ground-state wave function, which nowadays are called *quasi-particles*, so they are characterized by effective parameters such as an effective mass and a lifetime (yes, electrons *interacting* among themselves and with the lattice have also a finite lifetime, which is very closely related to their interaction probability; for more details on this subject see [12]), analogously to what we have said about Copper pairs in §2.3.1. Thus to understand deeply what is going on in tunnelling phenomena,

[1] I. Giaever, Phys. Rev. Letters **5**, 147 (1960); **5**, 464 (1960).

[2] J. Nicol, S. Shapiro and P. H. Smith, Phys. Rev. Letters **5**, 461 (1960).

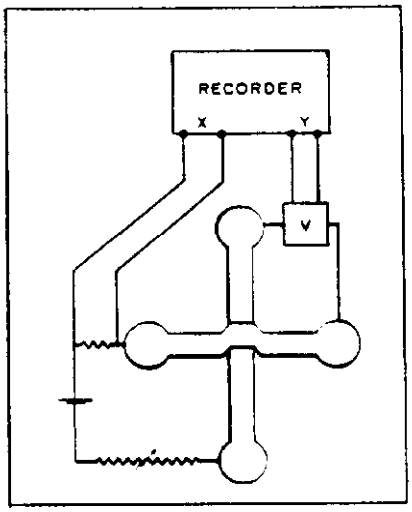

Figure 3.1: Scheme of a tunnelling junction.

the excitations should be studied in great detail and the excitation diagram should be found (for this appproach to tunnelling see [13] or [14, Chapter 2]). A simpler way to do it, which is the way I followed, is to make an analogy to what is done with semiconductors, this can be done because quasiparticles in the superconductivity theory, like electrons, are fermions and thus follow the Fermi-Dirac distribution [15, §1.12]. This is a simplicistic view, but it gives the correct results and it is much clearer to me; the only modification needed is to account for the superconducting density of states at 0 K as given by the equation

$$N_S(E) = N_N(E)\frac{E}{\sqrt{E^2-\Delta^2}}, \tag{3.1}$$

where $N_N(E)$ is the normal metal density of states and E is the energy of a quasiparticle, measured relatively to the Fermi energy. It is shown in [14, §1.04] that $N_N(E)$ looks like

$$N_N(E) = \frac{1}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)\sqrt{E}, \tag{3.2}$$

(where m_e is the electron mass) for a free electron model of a metal. We shall make the very adequate assumption that the normal metal density of states is essentially constant over the energy range of interest (typically less then 0.1 eV compared with a Fermi energy of several electron volt). It is readily seen that there is a singularity in the density of states at $\pm\Delta$, as measured from the Fermi energy. In Figure 3.2a the density of states for a superconductor at 0 K is shown. When the temperature is

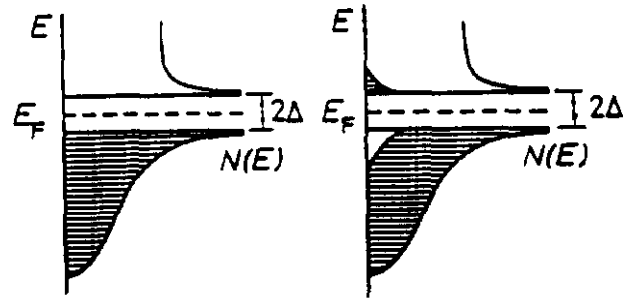

Figure 3.2: The density of states as a function of energy in the semiconductor representation. Note that the energy scale is very much distorted. The Fermi energy is by several orders of magnitude greater than the gap energy. (a) No excited states at zero temperature, (b) electrons above the gap at finite temperature.

greater than 0 K, some of the electrons are found over the energy gap, as is shown in Figure 3.2b. These graphs show the density of states (on the x axis) as a function of the excitation energy (on the y axis), the filled states are the dashed ones.

Let's consider, for example, a metal–insulator–metal junction [15, §2.2]. The energy required to remove a Fermi level electron from a metal is the work function $e\Phi$, and the electrons removed from two different isolated metals can be considered to be at the same energy. As the metals are brought together or connected through a circuit, electrons are transferred from one to the other, thus creating a potential difference $\Phi_1 - \Phi_2$ sufficient to eliminate the energy difference between the two Fermi levels, in thermal equilibrium. as shown in Figure 3.3a. If a negative voltage $-V$ is applied to the metal on the left-hand side[3] we will have a net electron flux going from the left-hand side of the junction, to the right-hand side of the junction. in such a way that energy is conserved (which means that electrons don't change their energy in the tunnelling process). The result for the tunnelling current is

$$I = AN_1(0)N_2(0)eV, \tag{3.3}$$

where $N_1(0)$ and $N_2(0)$ are the densities of states on the left-hand side and on the right-hand side of the junction at the Fermi energy, A is a coefficient incorporating both the probability of transition across the barrier and the geometry of the junction and $-e$ is the charge of the electron. So the result is that the relation between the current and the voltage is linear: a metal–insulator–metal junction follows the Ohm's

[3] All the junctions treated here behave indipendently of the sign of the applied voltage, therefore we will always consider the sign in such a way that the net electron flow is from left to right, following [15].

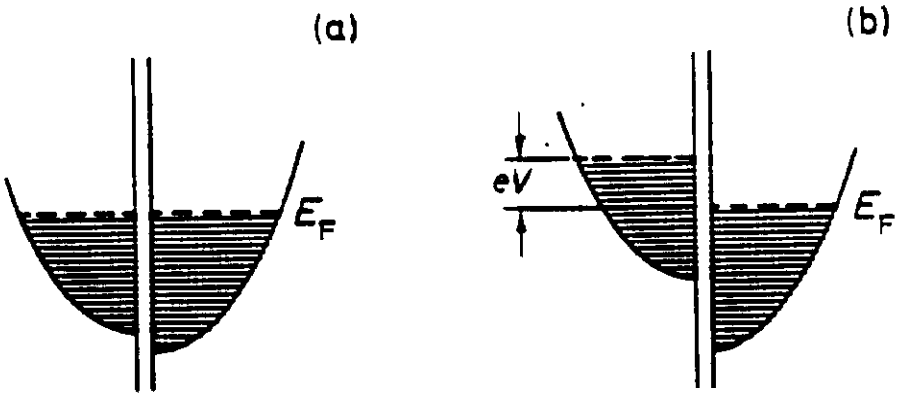

Figure 3.3: Energy diagram for a metal–insulator–metal junction (a) at thermal equilibrium, (b) when a potential difference V is applied.

law for small values of V [15, §2.2]. In what follows we will forget about the potential $-V$ being negative and the charge of the electron being $-e$, instead we will suppose to multiply together the two minus signes in order to get rid of them.

3.2 N–I–S junctions

We want now to turn considering the basic phenomena of tunnelling which occur in a junction when one of the two electrodes is a superconductor and the other one is a normal metal (the two electrodes are, of course, separated by a thin insulating film), such a junction is usually denoted N–I–S or simply N–S. We already now how a metal and a superconductor look like in the semiconductor representation, both at zero and finite temperature, let's try putting them together now.

At absolute zero temperature all the states are filled up to $E_F - \Delta$ and there are no filled states above the gap, in the superconductor and up to E_F in the metal. In thermal equilibrium the Fermi energies must match (Figure 3.4a). When we apply a voltage $V < \Delta/e$ the electrons on the left don't have access to empty states on the right and so there is no tunnelling current (Figure 3.4b) but when the voltage applied becomes greater than Δ/e, than there is a sudden onset of a current through the junction, not only because the energy of the electrons on the right is big enough to overcome the gap, but also because they face an enormous number of free states when they get to the superconductor side, as the density of states has a singularity at Δ. If we still increase the voltage, the first derivative of the I–V curve starts decreasing because the density of states on the right-hand side of the junction gets smaller at

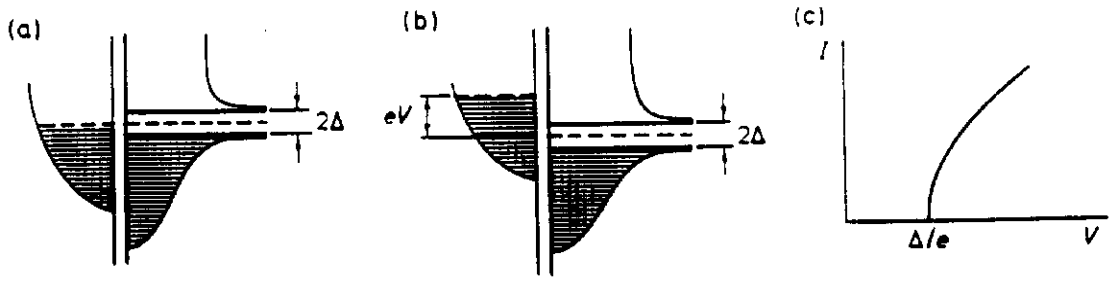

Figure 3.4: The energy diagram of an N–I–S junction in the semiconductor representation; (a) $V = 0$, (b) $V > \Delta/e$, (c) the I–V characteristics at $T = 0\,\mathrm{K}$.

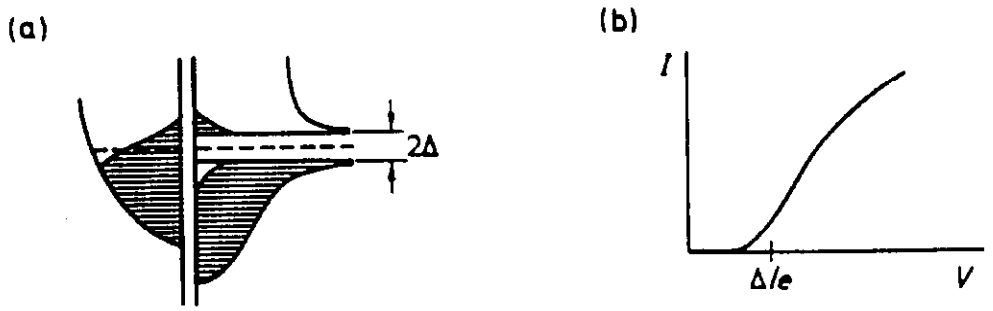

Figure 3.5: (a) The energy diagram of an N–I–S junction at finite temperature in thermal equilibrium, (b) the I–V characteristics at finite temperature.

higher energies (Figure 3.4c).

When we go to finite temperatures, some of the electrons in the metal have a sufficient energy to tunnel into the superconductor, even with a zero applied voltage (Figure 3.5a). In the superconductor too some of the states above the energy gap are occupied, leaving empty levels below the gap, thus contributing even more to the tunnelling current. So a very small voltage is sufficient to start a current through the junction, but a much more appreciable rise in the current is again found only when the biasing voltage is $V > \Delta/e$, as can be seen in the I–V characteristic in Figure 3.5b [15, §4.2].

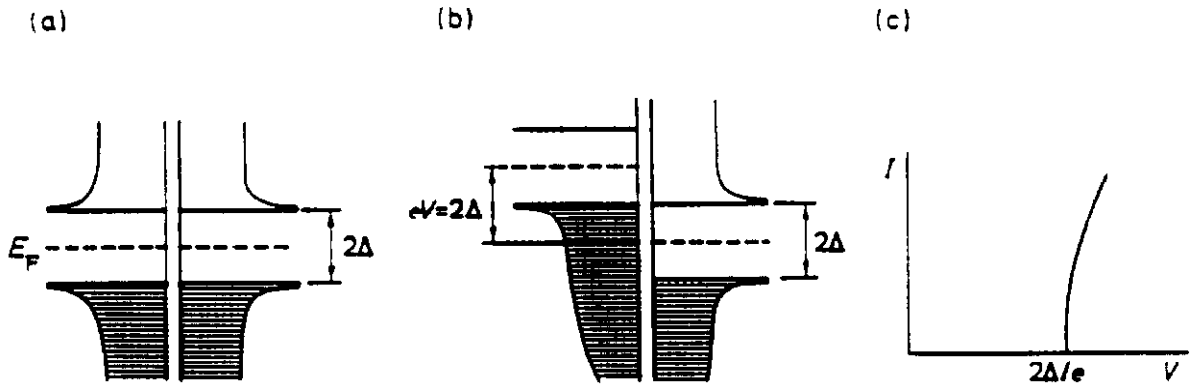

Figure 3.6: The energy diagram of an S–I–S junction; (a) $V = 0$, (b) $V = 2\Delta/e$. (c) the I–V characteristics at $T = 0\,\mathrm{K}$.

3.3 Quasiparticle tunnelling in superconductors

The next step is, of course, studying what happens when both sides of the junction are made of superconducting material (S–I–S junctions). We will first consider a junction between identical superconductors and then see what changes if the two superconductors are different, the parameter that must be accounted for the differences is, as we will see, the energy gap.

3.3.1 Junctions between identical superconductors

Let's, first of all, consider a junction between two identical superconductors at zero temperature (Figure 3.6a shows the energy diagram in the semiconductor representation). When no biasing voltage is applied to the junction, there is no current flowing, as all the energy levels are filled up to $E_F - \Delta$. The situation does not change as long as $V < 2\Delta/e$ because the the electrons below the gap on the left-hand side of the junction don't have access to empty states on the right-hand side. At $V = 2\Delta/e$ there is a sudden rise in current because there is plenty of free states on the right above the gap where the electrons from the left can tunnel (Figure 3.6b). At higher biasing voltages, the corresponding number of empty states is smaller (just as with the N–I–S junction), and so the current–voltage characteristic is the one shown in Figure 3.6c. When the temperature is different from absolute zero, there is an all overall rounding-off of the sharp features of Figure 3.6c which, of course, is very strongly temperature dependant. A set of measurement on an Al–Al_2O_3–Al junction made by

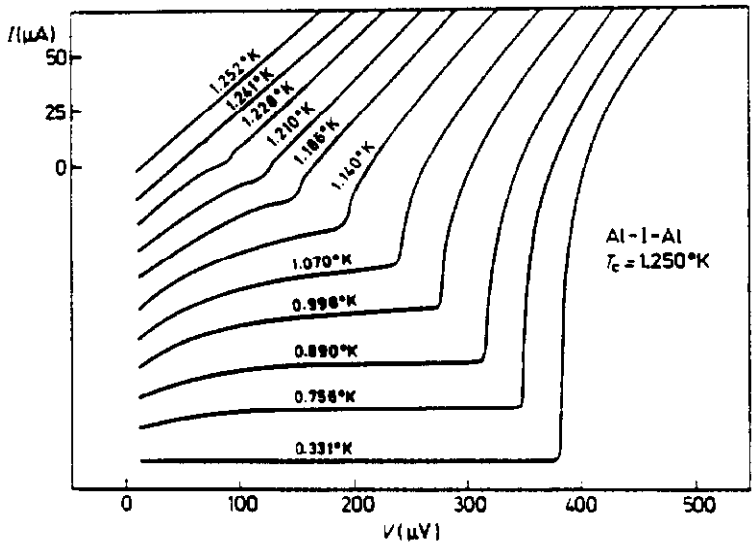

Figure 3.7: The I–V characteristics of an Al–I–Al junction (subsequent curves are displaced for clarity).

Blackford and March[4] is shown in Figure 3.7; it shows the temperature dependence of the current–voltage characteristic. At 1.252 K aluminum is in the normal state and the characteristic is linear, while at 1.241 K an effect of the energy gap is already visible [15, §4.3].

3.3.2 Junctions between different superconductors

At absolute zero temperature, the behaviour of the junction between two different superconductors is very similar to that of the junction between identical superconductors. In fact, electrons can tunnel from one side to the other only when there is a high enough biasing voltage, this occurs when $V = (\Delta_1 + \Delta_2)/e$, i.e. at twice the aritmetic mean between the two energy gaps (see Figure 3.8a). Thus, the I–V characteristics is very similar to that shown in Figure 3.6c, the only difference being the voltage at which there is the current onset, which is, for the junction between two different superconductors, $(\Delta_1 + \Delta_2)/e$ (see Figure 3.8b) [15, §4.4].

At finite temperature, we may still assume that the normal electron states above the larger gap are empty, but we cannot forget the thermally excited electrons in the smaller gap superconductor, as shown in Figure 3.9a for zero biasing voltage. Applying a voltage to the junction, will result in a rapidly increasing current because the thermally excited electrons on the left-hand side of the junction in Figure 3.9b can

[4]B. L. Blackford and R. H. March, Can. J. Phys., **46**, 141 (1968).

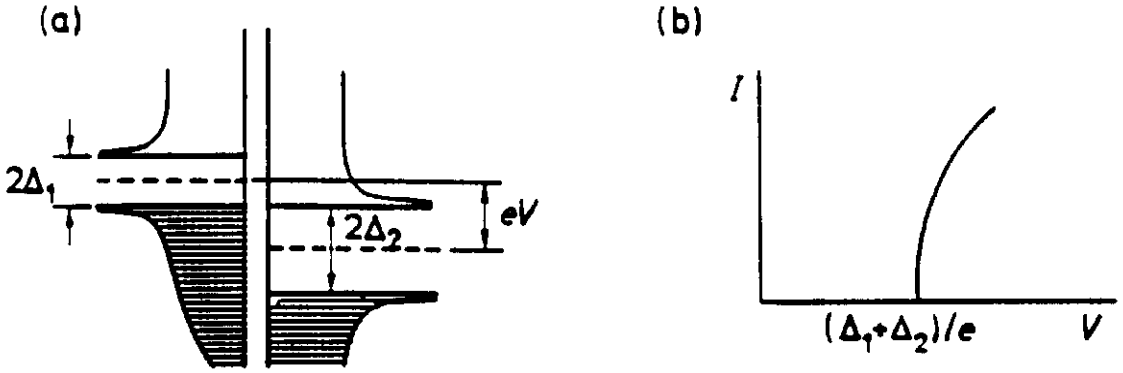

Figure 3.8: (a) Energy diagram and (b) I-V characteristics of an S_1–I–S_2 junction at $T = 0\,\mathrm{K}$.

tunnel to the right-hand side, thus producing a current. But when $V = (\Delta_2 - \Delta_1)/e$ is reached, all the thermally excited electron are facing empty states on the right-hand side of the junction, so a local weak maximum of the tunnelling current has been achieved. This maximum turns out to be a strong one, i.e. there exists a voltage range $\mathcal{V}$, containing $V_M = (\Delta_2 - \Delta_1)/e$, such that $I(V) < I(V_M)$ for every $V \in \mathcal{V}$, in fact the density of empty states decreases if we increase the voltage, so the total tunnelling current (which is proportional to the integral over the energy range in which electrons and empty states are simultaneously present of the product of the density of electrons, which is approximately constant in our case, the density of empty states, which is strongly dependent on the integration interval, and the tunnelling probability [15, §4.5]) decreases, as is shown in Figure 3.9c. The decrease in current continues until $V = (\Delta_1 + \Delta_2)/e$. At this point the electrons from below the gap of the left-hand side superconductor gain access to empty states on the right and this gives a very sudden increase in the current through the junction (Figure 3.9d). Thus the junction exhibits a *negative resistance* behaviour in the range

$$\frac{\Delta_2 - \Delta_1}{e} < V < \frac{\Delta_2 + \Delta_1}{e}. \tag{3.4}$$

If we apply even greater voltages to the junction, the voltage–current characteristic resembles very much the one for identical superconductors, as the differences in the energy gaps become negligible (Figure 3.9e).

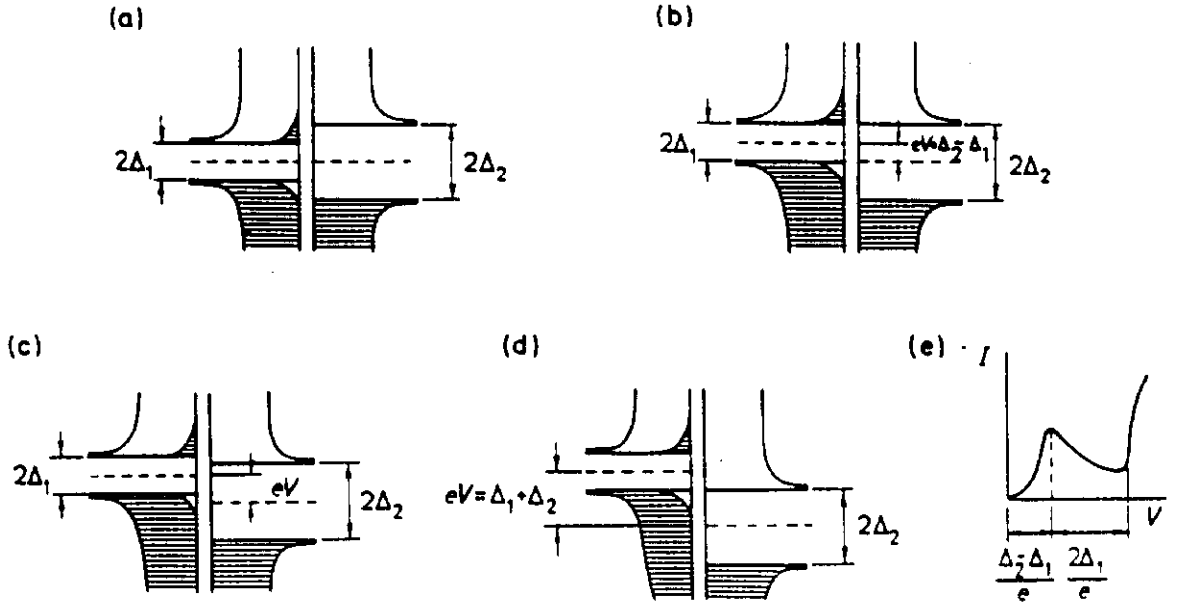

Figure 3.9: The energy diagrams and I–V characteristics of an S_1–I–S_2 junction at finite temperature; (a) $V = 0$, (b) $V = (\Delta_2 - \Delta_1)/e$, (c) $(\Delta_2 - \Delta_1)/e < V < (\Delta_2 + \Delta_1)/e$, (d) $V = (\Delta_2 + \Delta_1)/e$, (e) the I–V characteristics.

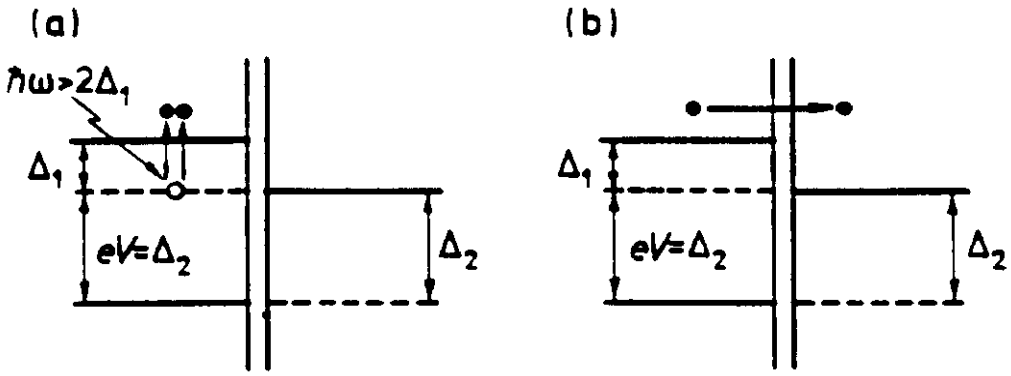

Figure 3.10: Effect of an incident photon on a tunnel junction; (a) a photon creates two electrons by breaking up a Cooper pair, (b) one of the electrons created tunnels across.

3.4 Photon-assisted tunnelling

We want to see now what happens if a tunnel junction is exposed to electro-magnetic radiation, together with a biasing voltage, and see if the I–V characteristics is modified in any way by this [15, §5.2].

First of all we will have to modify slightly the semiconductor-type diagram to account for the Cooper pairs, all condensed at the Fermi energy. The electron pairs will be represented by an empty circle.

The tunnelling current may be modified by illuminating the junction with electro-magnetic waves. It is easy to see that if the photon energy is greater than $2\Delta_1$, they will break up Cooper pairs and create two electrons above the gap as shown in Figure 3.10a, which can then tunnel across the barrier, as their number excedes the equilibrium value (Figure 3.10b), creating an extra current.

An influence on the tunnelling process is still possible then if the photon acts jointly with the applied voltage. Let's consider the situation at absolute zero temperature for the sake of simplicity, and analyze the case when $V = (\Delta_1 + \Delta_2)/e$. Then a Cooper pair may break up into two electrons, one of them may tunnel across the barrier, as is shown in Figure 3.11. If $V < (\Delta_1 + \Delta_2)/e$ then there is no tunnelling current, because the electrons originating from a Cooper pair breaking up cannot tunnel as, for energy to be conserved, the electron leaking to the right hand side of the junction should go below the energy gap, where, as we assumed to work at absolute zero temperature, there are no empty states (Figure 3.12a). However, if a photon of the right energy is available, then the tunnelling could take place through the mechanism

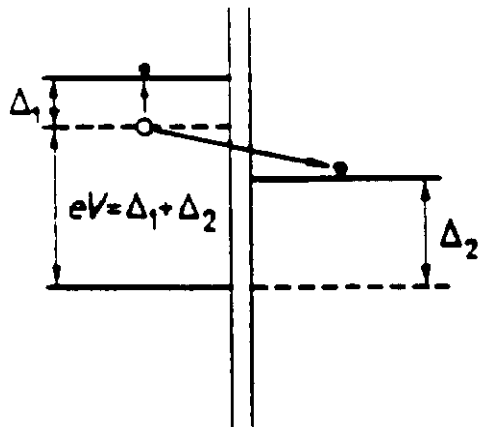

Figure 3.11: Excess tunnelling current in an S_1–I–S_2 junction due to the break up of a Cooper pair ($V = (\Delta_1 + \Delta_2)/e$).

suggested in Figure 3.12b, where the electron can the reach the top of the energy gap adsorbing a photon, and we would refer to the phenomenon as *photon-assisted tunnelling*. The mathematical relation for the onset of tunnelling current is

$$\hbar\omega = \Delta_1 + \Delta_2 - eV, \tag{3.5}$$

so that $\hbar\omega$ is just the missing energy to satisfy the energy conservation condition. If the energy of the photon is above this value, the tunnelling is still possible but it is less likely to happen, because the density of empty states at higher energy is smaller.

Still another possibility is that the energy of the photon is below the value given in equation (3.5), in this case tunnelling is still possible, provided multi-photon absorption processes are allowed. For example, referring to Figure 3.13a, an electron absorbing three photons simultaneously may tunnel across the barrier. Hence we may generalize equation (3.5) as follows

$$n\hbar\omega = \Delta_1 + \Delta_2 - eV, \tag{3.6}$$

thus expecting sudden rise in the tunnelling current whenever photons of the right frequencies are present in the voltage range $0 < V < (\Delta_1 + \Delta_2)/e$. When $V > (\Delta_1 + \Delta_2)/e$ we know that a current will flow even in the absence of electro-magnetic radiation, but, if photons of the right frequency are available, they can assist the tunnelling, as shown in Figure 3.13b. The mechanism works more or less like this, a Cooper pair breaks up and one of the electrons goes into a state just above the energy gap on the left-hand side. The other electron could tunnel to the superconductor on

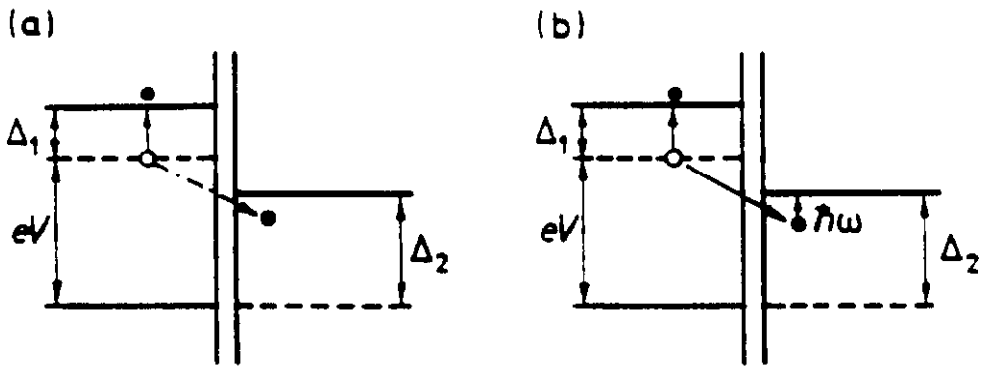

Figure 3.12: (a) Tunnelling not allowed; (b) tunnelling allowed if assisted by a photon.

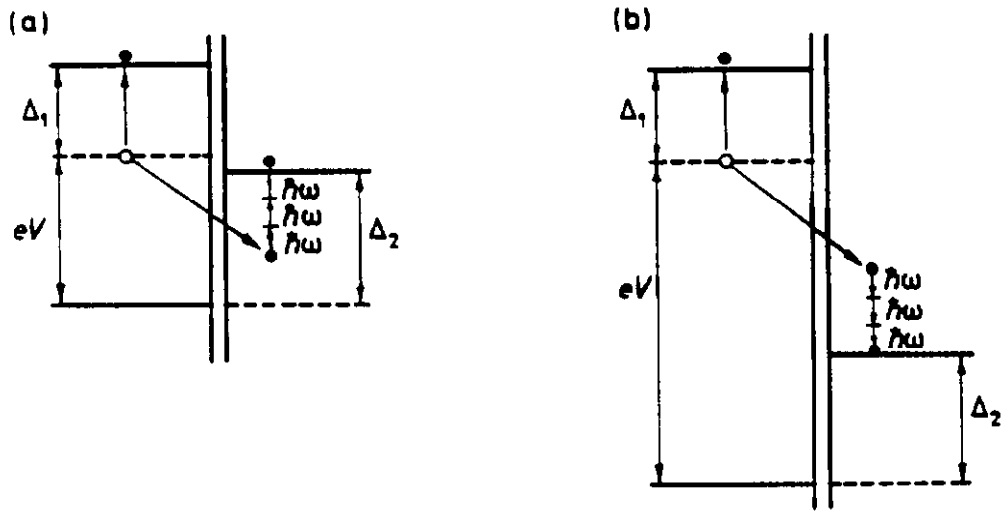

Figure 3.13: Tunnelling assisted (a) by absorption of three photons, (b) by emission of three photons.

the right well above the energy gap, to conserve the total energy, thus going into a region with a low density of empty states. This process would occur with much higher probability if the electron could tunnel into a region with a high density of empty states. This can happen if it can emit some photons (in our case three) at the same time and, being the photons Bose particles[5], the chance for this happening is greatly enhanced if there are many other photons with the same energy [3, chapter 4]. Thus the mechanism for tunnelling current increase is photon emission stimulated by input photons and, for an n-photon emission process the current rises occur when

$$V_n = \frac{1}{e}(\Delta_1 + \Delta_2 + n\hbar\omega). \tag{3.7}$$

The first experiments on tunnel junctions in the presence of electromagnetic waves were performed by Dayem and Martin[6]using junctions between Al and In. The frequency of the electro-magnetic waves employed was 38.83 GHz so the experimental solution was to place the sample inside a cavity, the results are reported in Figure 3.14, were the rises in current at multiples of $\hbar\omega/e$ are clearly visible.

[5] I found very useful, as an introduction to Bose particles [3, Chapter 4].

[6] A. Dayem and R. J. Martin, Phys. Rev. Letters 8, 246 (1962)

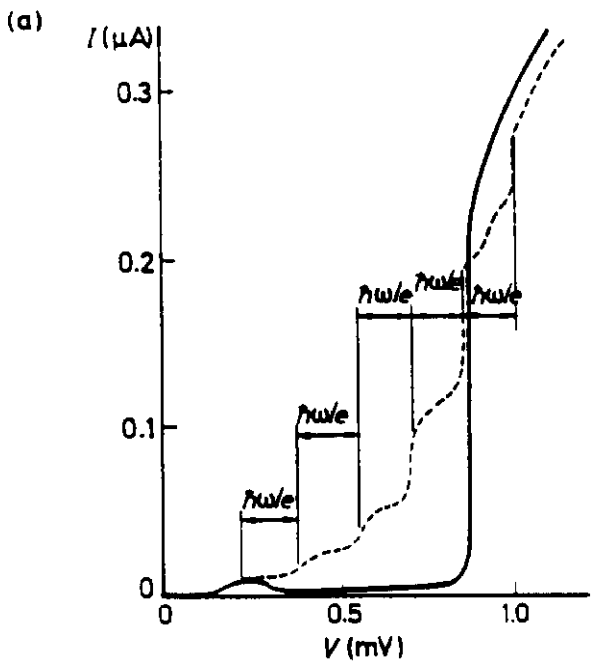

Figure 3.14: I–V characteristics of an Al–I–In junction in the absence (solid lines) and presence (dotted lines) of microwaves of frequency 38.83 GHz.

Chapter 4

The Josephson junction

4.1 Cooper pair tunnelling

The tunnelling processes discussed in chapter 2 involve the tunnelling of one or two quasiparticles from one side of the junction to the other. Josephson[1], in 1962, predicted additional tunnelling currents when both sides of the junction are superconducting, due to the direct leakage of Cooper pairs from one side of the barrier to the other (Figure 4.1). Unlike quasiparticle tunnelling, pair tunnelling does not involve excitations and can occur even without bias across the junction. Thus one could connect a current source to a junction and, for currents less than a certain critical value, no voltage would be developed if the current were carried across the insulator by Cooper pairs [14, page 140].

We will be talking of "barriers" or "insulators" throughout this chapter as we will deal mainly with Josephson oxide-barrier tunnel junctions, but we have to remember that there is a larger family of Josephson junctions which have as a common feature a weak connection between two superconducting regions; for example, a point contact junction is shown in Figure 4.2 (in one possible operating configuration).

To understand the Josephson effect, we can consider two pieces of superconducting material, initially separated. The absolute values of the phases of the microscopic wave function, in each of the two pieces, are arbitrary, since they are unobservable, but the *relative values* are fixed — in fact every Cooper pair in one piece must have the phase perfectly correlated with all the others. In fact, this statement is equivalent to the requirement for 0 total center of mass momentum for all Cooper pairs in the BCS ground state, as we have seen at the end of section 2.2 (remember that $\hbar\nabla\theta$ is the canonical momentum in an electro-magnetic field). The phases in the two pieces of superconducting material are uncorrelated, as long as they are kept separated [7,

[1]B. D. Josephson, Phys. Letters **1**, 251 (1962).

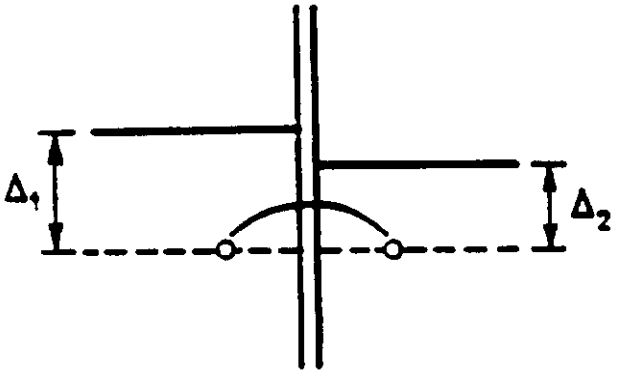

Figure 4.1: A Cooper pair tunnels across the junction in the absence of a potential difference.

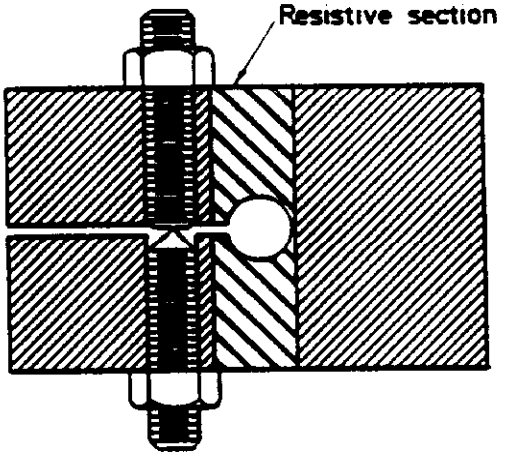

Figure 4.2: A possible realization of a point contact junction.

§11.11]. Consider now the extreme opposite, i.e. a very thin barrier, as its thickness tends to zero we approach a situation in which there is no barrier at all. In the absence of a barrier the macroscopic wave function $\tilde{\psi}$ must change continuously from one point to the other.

We see that as the barrier goes from very thick to very thin, the system loses one degree of freedom, corresponding to the ability of fixing arbitrarily the absolute value of the phase of $\tilde{\psi}$ on each of the sides of the barrier, as it has just been described. By what mechanism can this loss take place ? The free energy of the system contains a contribution from the barrier region which depends on the relative phases of the values of $\tilde{\psi}$ on the two sides of the barrier, and whose magnitude becomes greater as the barrier is made thinner (see [16] for more details). With very thick barriers the free energy contribution from the barrier is negligible and the phases can vary independently. When the barrier is made thinner, however, this contribution grows bigger and bigger (diverging in the zero barrier mass limit) thus fixing the phase relationship on the sides of the barrier.

Hence a current of pairs will flow across the barrier so as to lock the phases as soon as the coupling energy becomes comparable to kT. This happens when the barrier is very thin (~ 10 Å) and was first observed by Anderson and Rowell[2]. Because the junction with such a thin barrier behaves like a single superconductor, Anderson has called this "weak" superconductivity. We shall see that pair tunnelling can also take place when there is a voltage; in that case the phases of the wave functions are not locked together but rather slip relative to each other at a rate that is precisely related to the voltage.

4.2 The equations of the junction

In order to analyze the behaviour of the junction in a more quantitative way we will follow the "translation" that Feynman did of the Josephson effect in terms of coupled two-states systems [3, §21-9].

We will call the amplitude to find the Cooper pair on the left-hand side of the junction $\tilde{\psi}_1$, and the amplitude to find it on the right-hand side $\tilde{\psi}_2$. We will be considering, for the sake of simplicity, a junction between identical superconductors. The two amplitudes should then be related in the following way

$$i\hbar\frac{\partial\tilde{\psi}_1}{\partial t} = U_1\tilde{\psi}_1 + K\tilde{\psi}_2, \tag{4.1}$$

$$i\hbar\frac{\partial\tilde{\psi}_2}{\partial t} = U_2\tilde{\psi}_2 + K\tilde{\psi}_1. \tag{4.2}$$

[2]P. W. Anderson and J. M. Rowell, Phys. Rev. Letters 10, 230 (1963).

The constant K is the energy term that expresses the correlation between the two sides of the junction. If $K = 0$ these two equations would just describe the ground states of the left and right-hand side of the junction, but if $K \neq 0$, then there is a finite probability for a pair tunnelling from one side to the other. Let's assume now that a voltage source is applied between the two sides of the junction; a difference of energy $q^*(V_2 - V_1) = q^*V$ is imposed between the two sides so that

$$U_2 - U_1 = q^*V. \tag{4.3}$$

For convenience we can set the zero of energy, halfway between the energies U_1 and U_2, so that the coupled Schrödinger equations of the junction become

$$i\hbar\frac{\partial\tilde{\psi}_1}{\partial t} = \frac{q^*V}{2}\tilde{\psi}_1 + K\tilde{\psi}_2, \tag{4.4}$$

$$i\hbar\frac{\partial\tilde{\psi}_2}{\partial t} = -\frac{q^*V}{2}\tilde{\psi}_2 + K\tilde{\psi}_1. \tag{4.5}$$

Let's now make the substitutions:

$$\begin{cases} \tilde{\psi}_1 \stackrel{\text{def}}{=} \sqrt{\rho_1}e^{i\theta_1}, \\ \tilde{\psi}_2 \stackrel{\text{def}}{=} \sqrt{\rho_2}e^{i\theta_2}, \end{cases} \tag{4.6}$$

where ρ and θ have the same meaning we gave them in section 2.2. I will not go through the details of the algebra, I will just give the results, which can be obtained separating the imaginary and real parts of the equations, letting $\delta \stackrel{\text{def}}{=} \theta_2 - \theta_1$:

$$\dot{\rho}_1 = +\frac{2}{\hbar}K\sqrt{\rho_2\rho_1}\sin\delta, \tag{4.7}$$

$$\dot{\rho}_2 = -\frac{2}{\hbar}K\sqrt{\rho_2\rho_1}\sin\delta, \tag{4.8}$$

$$\dot{\theta}_1 = -\frac{K}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}}\cos\delta + \frac{q^*V}{2\hbar}, \tag{4.9}$$

$$\dot{\theta}_1 = -\frac{K}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}}\cos\delta - \frac{q^*V}{2\hbar}. \tag{4.10}$$

From (4.7) and (4.8) we see that the rate of decrease of the pair density in one superconductor is the negative of that in the other. This rate of change represents only a tendency to change, as a real change would create a charge imbalance between the electrons and the background of positive ions, which in turn generates very strong restoring Coulomb forces. This imbalance is avoided by currents flowing in the circuit

connected to the junction [14, page 141]. These Cooper pairs flowing from side 1 to side 2. create a current density $\mathbf{J}_s$ directed from 2 to 1 whose magnitude is equal to

$$J_s = \frac{2K}{\hbar}\sqrt{\rho_1\rho_2}\sin\delta. \tag{4.11}$$

and from what we have said before, the product $\sqrt{\rho_2\rho_1}$ can be set equal to ρ_0. The other pair of equations (subtracting (4.9) from (4.10) and equating $\rho_1 = \rho_2$) tells us

$$\dot{\delta} = \frac{2e}{\hbar}V, \tag{4.12}$$

where we have used $q^* = -2e$.

We have thus found the two fondamental equations for a Josephson junction

$$J_s = J_c \sin\delta. \tag{4.13}$$

$$\frac{\partial\delta}{\partial t} = \frac{2e}{\hbar}V. \tag{4.14}$$

We will now use these equations to explain the experimentally observed effects mentioned above.

4.3 The Josephson effects

4.3.1 The d.c. Josephson effect

The equations derived in the previous section become considerably simpler for the time-independent case, giving rise to the so called *d.c. Josephson effect*. It follows from equation (4.14) that if the relative phase is independent of time then $V = 0$. Thus, if we have these two coupled superconductors, so that the phases are strongly correlated and constant over the whole junction (this is true if there aren't magnetic fields, as we will see), we are left only with equation (4.13). This equation tells that if $\delta \neq 0$ (and there is no particular reason why δ should be equal to zero), then there is a finite current flowing through the insulator, and this can happen without causing any voltage drop. So, the insulator behaves just like a superconductor. From equation (4.13) follows also that J_c is the maximum current that can go through the junction without having a voltage drop. So, for any value of J_s below the critical current density J_c, the phase difference between the two superconductors is determined by the external circuit [15, §10.1].

Let's now try to investigate what happens when the external circuit is made by an ideal current generator and we force a current density $J_s > J_c$ through the junction. In the simplest case the voltage will jump from Josephson tunnelling characteristic to

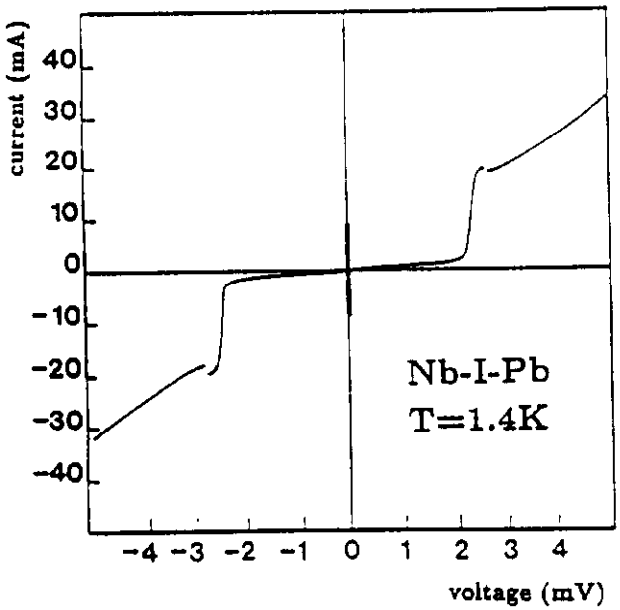

Figure 4.3: The I–V characteristics of a Nb–I–Pb Josephson junction clearly showing the Josephson current at zero voltage.

the normal S–I–S tunnel junction characteristic, as shown in Figure 4.3. The supercurrent, its maximum value, the transition from the Josephson junction characteristic to the normal S–I–S junction were observed for the first time by Anderson and Rowell in the 1963 paper (cited in footnote 2), only nine months after Josephson's original prediction.

4.3.2 The a.c. Josephson effect

The a.c. Josephson effect—as the name implies—is concerned with the temporal variation of the phase across the junction. All the relevant properties can be inferred from equation (4.14), but we will just see an approximate solution to have some physical insights and will not bother about a more general treatment (the interested reader can find it in [15, §10.2]).

The simplest case is when a d.c. voltage V_0 is applied through the junction, in fact integration of (4.14) over the junction gives

$$\delta = \delta_0 + \frac{2e}{\hbar} V_0 t. \tag{4.15}$$

If we substitute this equation in (4.13), and integrate the whole thing over the junc-

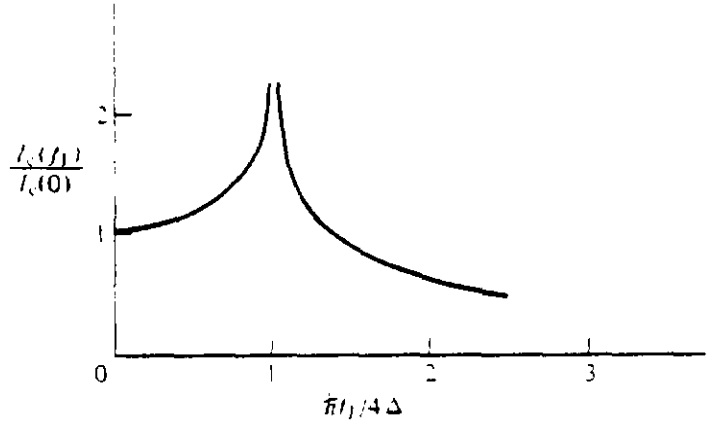

Figure 4.4: Dependence of the critical current I_c on the Josephson frequency, the peak occurs where the applied d.c. voltage is $2\Delta/e$.

tion's cross section, we find

$$I_s = I_c \sin(w_J t + \delta_0), \tag{4.16}$$

and the frequency of the a.c. current is (Josephson frequency)

$$f_J \stackrel{\text{def}}{=} \frac{\omega_J}{2\pi} \stackrel{\text{def}}{=} \left(\frac{1}{2\pi}\right) \frac{2e}{\hbar} V_0. \tag{4.17}$$

The coefficient in the last term is 483.6×10^{12} Hz/V. It has also been shown that I_c is frequency dependent, thus the amplitude of the oscillating current varies with frequency (the dependence is sketched in Figure 4.4 and it has been verified experimentally) and the result is that a substantial a.c. Cooper pair current flows even when the voltage (and thus the frequency) exceeds the gap by several times [14, page 144].

4.4 Effect of a magnetic field

We have seen in Chapter 2 that the current flowing through a superconductor is directly related to the vector potential via equation (2.5). If we require that equation to be gauge invariant under a vector potential gauge transformation (see [17, §6.5] for a review on gauges and gauge transformations in electro-magnetism),

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi, \tag{4.18}$$

we find:

$$\theta \to \theta - \frac{2e}{\hbar}\chi. \tag{4.19}$$

If we now consider the junction between the pieces of superconducting materials we have the two relations

$$\begin{cases} \theta_1 & \to & \theta_1 - (2e/\hbar)\chi_1, \\ \theta_2 & \to & \theta_2 - (2e/\hbar)\chi_2. \end{cases} \tag{4.20}$$

We are now able to write a gauge invariant phase difference. Using (4.18) and (4.20) we find (remembering that $\int_1^2 \nabla\chi \cdot d\mathbf{l} = \chi_2 - \chi_1$),

$$\tilde{\delta} \stackrel{\text{def}}{=} \theta_2 - \theta_1 + \frac{2e}{\hbar}\int_1^2 \mathbf{A} \cdot d\mathbf{l}. \tag{4.21}$$

We have thus found how a vector potential creates a shift in the phase difference.

It is worth noting that in our equations, we always deal with the vector potential instead of the magnetic induction **B**. This isn't a new fact as we know that quantum mechanics is sensitive to potentials (this is the Aharonov-Bohm effect, for an interesting discussion see [18, §15-5] or the original papers [19, 20]), the main idea behind this is that the behaviour of a quantum mechanical system is completely determined if we know its Hamiltonian, which is written in terms of the potentials, rather than the fields, as it represents the total energy of the system[3] [14, §4.03].

4.5 The general Josephson junction

4.5.1 The shunted junction model

Let's now consider the most general Josephson junction, and try to understand its behaviour. In section 4.2 we have derived the equations for an oxide-barrier tunnel junction and, in section 4.4, we have modified them to make them gauge invariant, upon integration over the junction surface they finally read

$$I_s = I_c \sin\tilde{\delta}, \tag{4.22}$$

$$\frac{\partial\tilde{\delta}}{\partial t} = \frac{2e}{\hbar}V. \tag{4.23}$$

It should be clear that these two equations describe only the current carried by electron pairs (that's why I always put the "s" subscript), so we want to understand

[3]This is not always true, if we consider a classical system whose constraints are time-dependent, the Hamiltonian does not represent anymore the total energy, at least in an inertial frame (it might, for example, represent it in a rotating frame so that it would contain also some energy term coming from the inertial forces).

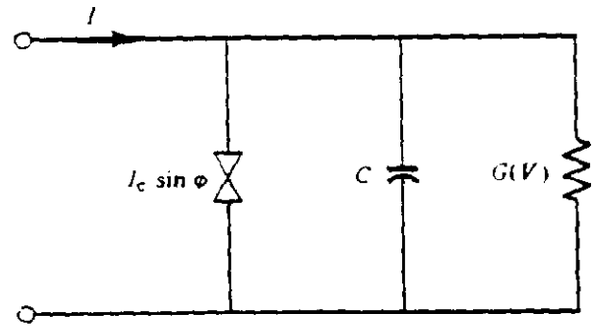

Figure 4.5: Equivalent circuit for a Josephson junction.

which are the other contributions to the total current. To normal electrons, the oxide-barrier junction is very much like a capacitor, so in general displacement currents might be important. We have also seen that when $V \neq 0$ there is a quasiparticle tunnelling current in parallel to the other terms, and finally, if the oxide is not a perfect insulator there will also be a regular electron current [14, §5.02].

We can thus draw an equivalent circuit for a general real Josephson junction, whose parameters will fix the type and characteristics of the junction. The equivalent circuit is shown in Figure 4.5, where we have drawn a generic voltage-dependent conductance $G(V)$ to account for the quasiparticle tunnelling and the insulator leakage currents, and a capacitor for the displacement current. The voltage dependence $G(V)$ can be found from the I–V characteristics of the junction and in general it will not be a simple dependence, so the solution of this equivalent circuit could be a very complicate numerical business.

Let's suppose, in a first and very crude approximation, that

$$G(V) = G, \tag{4.24}$$

and let's also assume that the circuit is driven via a d.c. current source (with infinite internal impedance). If there is no vector potential $\mathbf{A}$ in the circuit area[4] then $\tilde{\delta}$ is only a function of time and the equation of the circuit is

$$I = I_c \sin\tilde{\delta} + GV + C\frac{dV}{dt}. \tag{4.25}$$

[4]If $\mathbf{A} = 0$ everywhere in some fixed gauge, then it will be equal to a gradient, in any other gauge, and this just displaces the phase of the wave function, not creating any physically observable difference (for example a current), thus preserving the gauge invariance of the equations.

Let's replace the time variable with the adimensional variable τ defined as

$$\tau \stackrel{\text{def}}{=} \frac{2e}{\hbar}\frac{I_c}{G}t, \tag{4.26}$$

thus finding

$$\frac{I}{I_c} = \beta_c \frac{d^2\tilde{\delta}}{d\tau^2} + \frac{d\tilde{\delta}}{d\tau} + \sin\tilde{\delta}, \tag{4.27}$$

where

$$\beta_c \stackrel{\text{def}}{=} \frac{2e}{\hbar}\frac{I_c}{G}\frac{C}{G}. \tag{4.28}$$

Our aim is to find the average value of the voltage

$$V = \left\langle \frac{\hbar}{2e}\frac{d\tilde{\delta}}{dt} \right\rangle, \tag{4.29}$$

(where the angle brackets stand for the time average) with a given applied constant current I. The simplest case is the one with $C = 0$, so that $\beta_c = 0$, the equation of the circuit can be solved exactly and the result is

$$V(I) = \begin{cases} 0 & \text{if } I < I_c \\ (I_c/G)[(I/I_c)^2 - 1]^{1/2} & \text{if } I \geq I_c, \end{cases} \tag{4.30}$$

the solution is plotted in Figure 4.6, it is readily seen that the solution is one-valued everywhere.

The situation becomes more complex when we consider $\beta_c \neq 0$. For $I > I_c$ there is again one value of the voltage for each value of the current but for the range $I_{\text{min}} \leq I \leq I_c$ there are two values of the voltage, one at $V = 0$ and the other at $V \neq 0$ (Figure 4.7). If a source with $\beta_c \neq 0$ was connected to a d.c. source and the current raised from zero, the I–V characteristics would be traced out as shown by the arrows, clearly showing *an hysteretic behaviour.*

Let's try to see what happens as $\beta_c \to \infty$, i.e. as the displacement current becomes the most important term in the equation of the circuit (4.27). The fact that $\beta_c \to \infty$ is equivalent to $C \to \infty$, keeping G fixed. So for the displacement current to remain finite, we have

$$I_d = C\frac{dV}{dt}, \tag{4.31}$$

so that we find

$$\frac{dV}{dt} = 0. \tag{4.32}$$

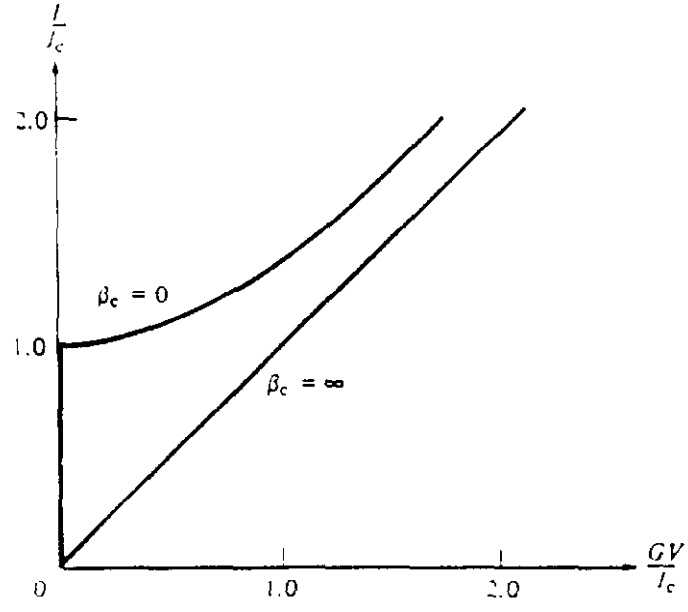

Figure 4.6: Normalized I–V characteristics for a Josephson junction described by the equivalent circuit of Figure 4.5 for cases of negligible ($\beta_c = 0$) and dominating ($\beta_c = \infty$) capacitance.

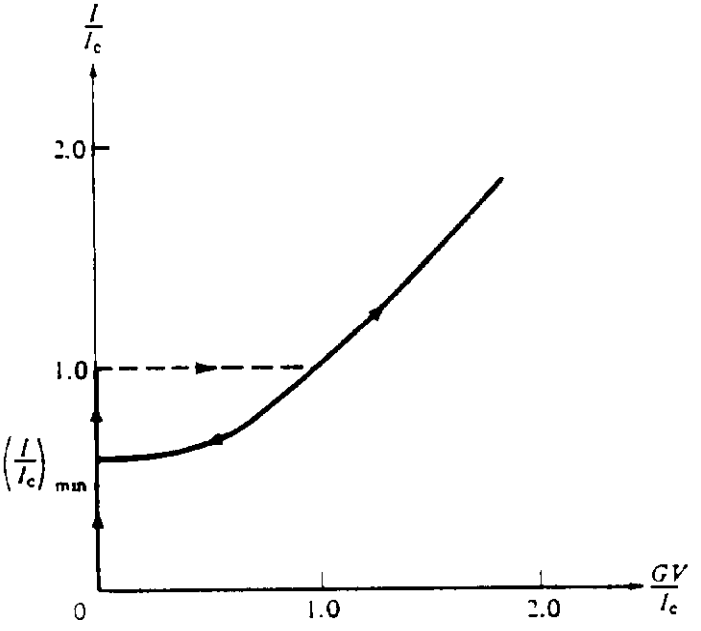

Figure 4.7: Normalized I–V characteristics for a Josephson junction represented by the equivalent circuit of Figure 4.5 with $\beta_c = 4$.

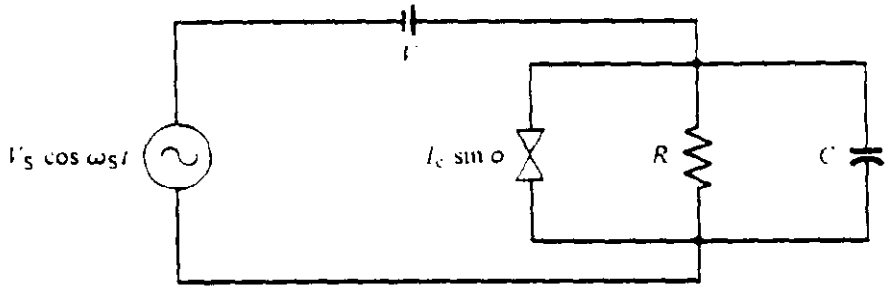

Figure 4.8: Equivalent circuit for a Josephson device connected to d.c. and a.c. voltage sources.

This means that the term due to the electron pairs' current shunted through the capacitance has 0 time average (it is just the a.c. Josephson effect) when $V \neq 0$, so the total current is just

$$I = GV \quad \text{if } V \neq 0. \tag{4.33}$$

Thus we have found also the second curve displayed in Figure 4.6.

We note that the shape of the I–V characteristics for $\beta_c = \infty$ is quite different from that given in Figure 4.3 for an oxide-barrier tunnel junction, even if we would expect them to be very similar. The reason for this difference is that the shunt conductance is far from constant. Far better results can be obtained substituting the I–V characteristics in equation (4.27) and solving numerically.

4.5.2 An application: RF effects

We want to investigate what happens when we drive a Josephson junction simultaneously with both an a.c. and d.c. sources, which is the usual situation in RF detection. Typically the junction is connected to a d.c. bias source and the RF signal is impressed on the junction by placing it in a microwave waveguide or cavity [14, §5.05].

We wil approximate this situation using a d.c. and an a.c. voltage sources with no internal impedance, as shown in Figure 4.8. Taking the voltage to be $V + V_s \cos \omega_s t$ and using the shunted junction equivalent circuit with constant resistance R we can compute the Josephson current, using the relationships derived above

$$\begin{aligned} I_s(t) &= I_c \sin \left[\int_0^t (2e/\hbar) v(t')\, dt' + \delta_0 \right] \\ &= I_c \sin \left[(2e/\hbar) V t + (2e/\hbar)(V_s/\omega_s) \sin \omega_s t + \delta_0 \right]. \end{aligned} \tag{4.34}$$

If we assume that $V_s \ll V$, we can follow [3, 21-9] writing

$$\sin(x + \Delta x) \approx \sin x + \Delta x \cos x. \tag{4.35}$$

for Δx small compared to x, and using this approximation in the (4.34)

$$I_s = Ic\left[\sin\left(\frac{2eV}{\hbar}t + \delta_0\right) + \frac{2eV_s}{\hbar\omega_s}\sin\omega_s t\cos\left(\frac{2eV}{\hbar}t + \delta_0\right)\right]. \tag{4.36}$$

The first term has zero time average, but the second can be written as

$$\frac{1}{2}\left[\sin\left(\omega_s t - \frac{2eV}{\hbar}t - \delta_0\right) + \sin\left(\omega_s t + \frac{2eV}{\hbar}t + \delta_0\right)\right], \tag{4.37}$$

and so its time average is different from zero only when

$$V = \frac{\hbar}{2e}\omega_s, \tag{4.38}$$

so a current spike is found at this voltage. An exact calculation of the current as in [14, 5.05] gives

$$I_s(t) = I_c\sum_{n=-\infty}^{\infty}(-1)^n J_n\left(\frac{2eV_s}{\hbar\omega_s}\right)\sin[(\omega_J - n\omega_s)t + \delta_0], \tag{4.39}$$

where $\omega_J \stackrel{\text{def}}{=} 2eV/\hbar$ and $J_n(x)$ is the n^{th} order Bessel function of the first kind. From this result we see that at values of the voltage

$$V_n = \frac{\hbar\omega_s}{2e}n, \tag{4.40}$$

there are current spikes in the I–V characteristics and they have a maximum height of $I_c J_n[2eV/(\hbar\omega_s)]$, where the phase $\delta_0 = \pi/2$. For each value of the d.c. voltage there is also a d.c. current through the resistance $I_G = V/R$ (Figure 4.8).

Actual measurements do not reveal these spikes in the I–V characteristics, because the junctions usually have resistances low enough that the sources act more like current sources. As a result, increasing the d.c. current source from zero raises the operating point along a step-like locus as shown by the arrows in Figure 4.9. In Figure 4.10 are shown some experimental measurements with increasing RF power. the steps are clearly visible and (known ω_s) this is a very precise method to measure $\Phi_0 \stackrel{\text{def}}{=} h/(2e)$, the quantum of flux.

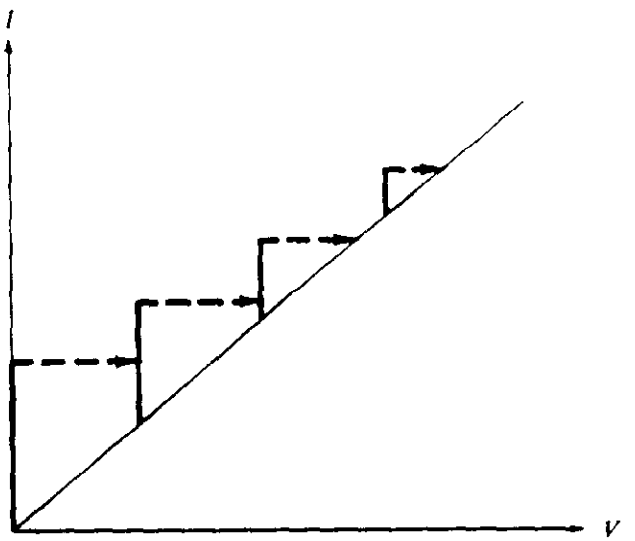

Figure 4.9: I–V characteristics for the circuit of Figure 4.8. In the voltage-source model, spikes in the d.c. current occur at voltages $V = n\hbar\omega_s/(2e)$. Measurements are usually made with sources that do not have zero impedances; steps, rather than spikes, are observed (broken lines).

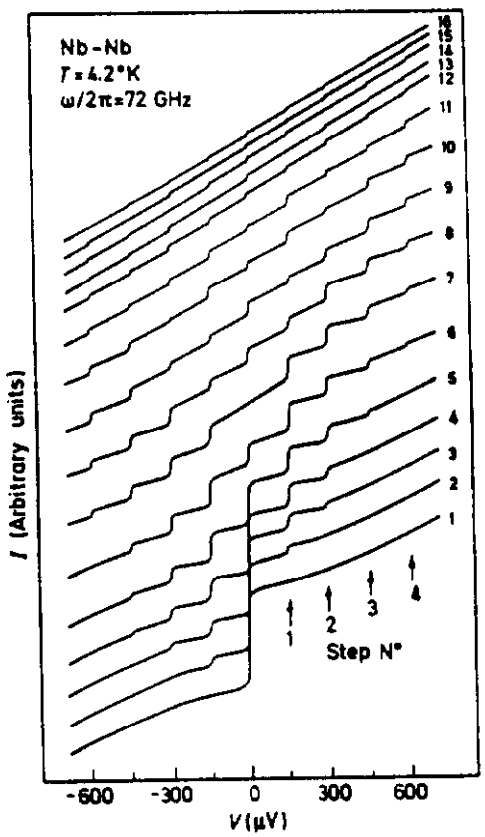

Figure 4.10: I–V characteristics of a Nb–Nb point contact junction taken by a high impedance source. 1: no microwave power, 2–16: microwave power increasing gradually by 26 db.

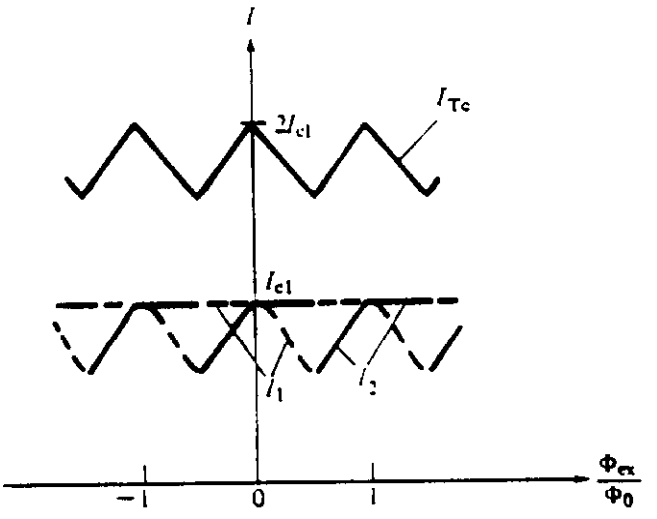

Figure 4.14: Total critical current and individual junction currents in a two-junction parallel symmetrical array with self-induced fields included.

where $I_{\text{circ}} = (1/2)(I_2 - I_1)$ so that the currents flowing through the single junction are $(1/2)I_T + I_{\text{circ}}$ and $(1/2)I_T - I_{\text{circ}}$ (because in one of the junctions both $(1/2)I_T$ and I_{circ} have the same direction, while in they other they have opposite directions [21, pages 34–35]). Equation (4.46) can then be expressed

$$\Phi_s = \frac{1}{2}L(I_2 - I_1) = \frac{1}{2}I_c(\sin\tilde{\delta}_2 - \sin\tilde{\delta}_1), \tag{4.47}$$

so that the total flux in the loop is

$$\Phi = \Phi_s + \Phi_{\text{ext}} = \Phi_{\text{ext}} + \frac{1}{2}LI_c(\sin\tilde{\delta}_2 - \sin\tilde{\delta}_1). \tag{4.48}$$

Solutions of this equation in terms of $\tilde{\delta}_1$ and Φ_{ext} allows determination of the total flux, and, therefore, of the total current as functions of $\tilde{\delta}_1$. For each value of the applied flux Φ_{ext}, the current I_T is maximized with respect to $\tilde{\delta}_1$, and the result is shown in Figure 4.14.

In Figure 4.15 is shown the total flux in the loop as a function of the applied flux Φ_{ext}. The deviation of this line from a 45° line are caused by Φ_s, the flux generated by circulating currents. At $(n+\frac{1}{2})\Phi_0$ the circulating current reaches a value for which a step increase is energetically advantageous. The circulating currents try, but fail to keep the flux in the loop at multiples of a flux quantum (because of the non-perfect

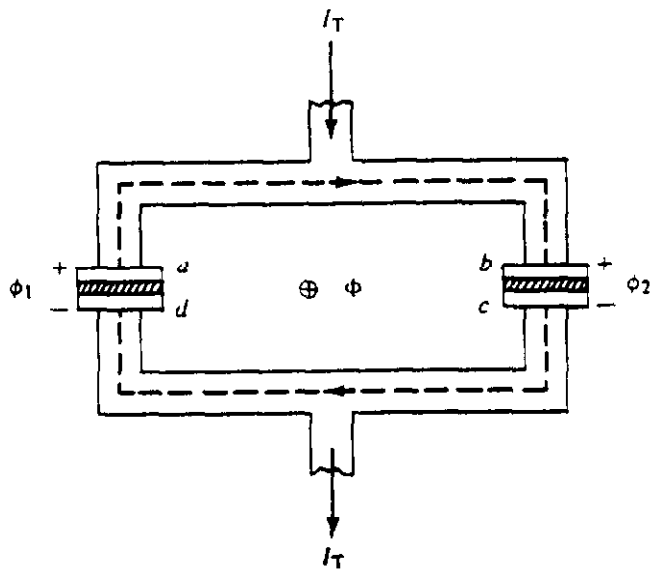

Figure 4.11: Two-junction parallel array with simmetrical feed. The integration path for the analysis is shown by the broken line.

4.6 Quantum interference in parallel junctions

4.6.1 General relationships

Let's consider now two Josephson junctions connected in parallel through superconducting paths, large enough so that the current density is zero inside the links joining the two junctions (from "a" to "b" and from "c" to "d") and therefore (in the London gauge) $\mathbf{A} = -\Lambda \mathbf{J}_s = 0$; the circuit described here is shown in Figure 4.11. Let I_T, I_1, I_2 be respectively, the external current and the currents circulating through junction 1 and 2 [14, §5.10]. Let's choose the London gauge—in which we have seen that $\nabla\theta = 0$, inside bulk superconductors (cf. section 2.3.3)—and take the integral of $\nabla\theta$ in the clockwise direction, the only surviving contributes are those from the two junctions, so

$$\oint \nabla\theta \cdot d\mathbf{l} = (\theta_a - \theta_d) - (\theta_c - \theta_b) = 2n\pi, \tag{4.41}$$

where the $2n\pi$ result comes from requiring the wave function to be one-valued everywhere. Let's now make use of the gauge invariant phase, defined in section 4.4, we

can write (4.41) as

$$\tilde{\delta}_2 = \tilde{\delta}_1 + \frac{2e}{\hbar}\oint \mathbf{A}\cdot d\mathrm{l} - 2n\pi. \tag{4.42}$$

We can disregard the $2n\pi$ because when we take the sine of the expression above, we will have $\sin(x - 2n\pi) = \sin x$. Using Stokes' theorem the above expression can be transformed into

$$\tilde{\delta}_2 = \tilde{\delta}_1 - \frac{2e}{\hbar}\Phi = \tilde{\delta}_1 - \frac{\Phi}{\Phi_0}, \tag{4.43}$$

where Φ is the magnetic flux inward through the loop, and $\Phi_0 \stackrel{\text{def}}{=} h/(2e)$ is the quantum of flux. The total current through the parallel junctions is

$$I_T = I_1 + I_2 = I_c\left[\sin\tilde{\delta}_1 + \sin\left(\tilde{\delta}_1 - 2\pi\frac{\Phi}{\Phi_0}\right)\right], \tag{4.44}$$

where we have supposed to deal with simmetric junctions.

4.6.2 Self-induced flux neglected

The analysis of the critical current is greatly simplified if we neglect the fact that currents flowing in the loop produce a flux themselves in addition to that of the externally applied field, Φ_{ext}. With these assumption, Φ becomes an independent variable and equation (4.44) can be maximized with respect to $\tilde{\delta}_1$, to find the critical current of the device, yielding

$$I_{Tc}(\Phi_{\text{ext}}) = 2I_c\left|\cos\left(\pi\frac{\Phi_{\text{ext}}}{\Phi_0}\right)\right|. \tag{4.45}$$

The dependence of the total critical current on the externally applied magnetic flux is shown in Figure 4.12.

Let's consider how the I–V characteristics is modified by the presence of a magnetic field. We assume that the junction is of the type with $\beta_c = 0$ (high conductance). When there is an integer number of flux quanta in the loop, the I–V characteristics is the same as the one we saw in section 4.5.1 and it is shown in Figure 4.13. When the external flux is $\Phi_{\text{ext}} = (n + \frac{1}{2})\Phi_0$ the two supercurrents interefere destructively and the I–V characteristics reduces just to the Ohm's law (because of the strong influence of the conductance).

4.6.3 Symmetrical array with self-induced flux

The definition of inductance requires the self-induced flux to be

$$\Phi_s = LI_{\text{circ}}, \tag{4.46}$$

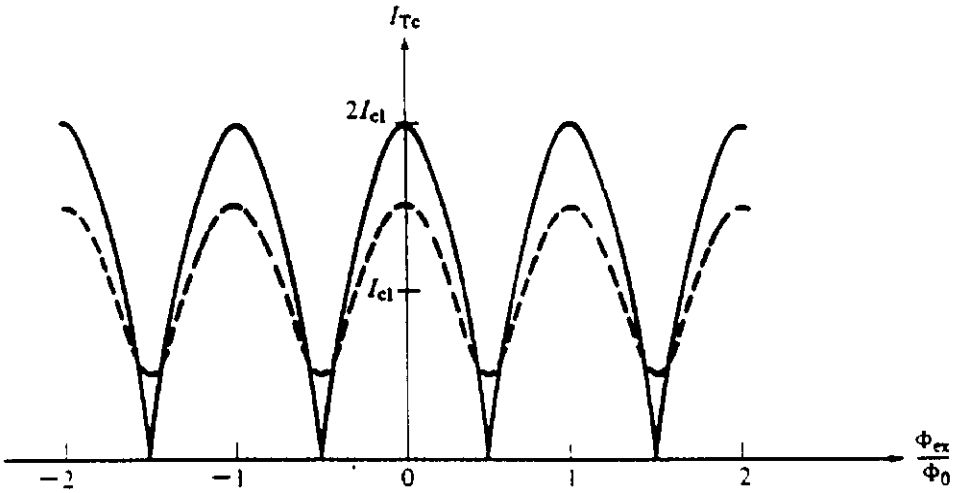

Figure 4.12: Dependence of the critical current for a two-junction parallel array on the applied flux where the self-induced flux is neglected. Solid line for symmetrical array and broken line for asymmetrical array with $I_{c1} = 2I_{c2}$.

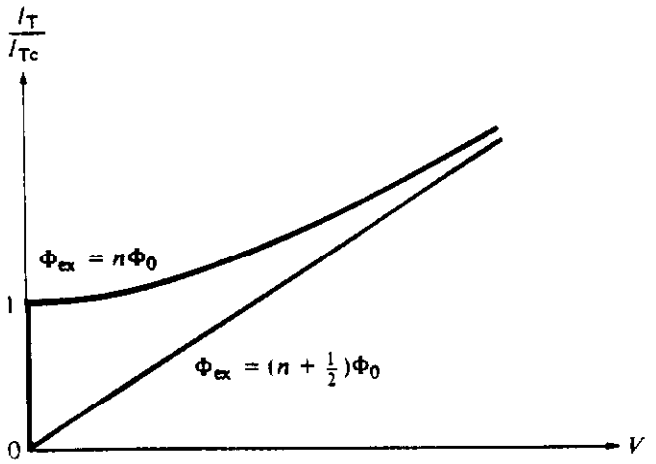

Figure 4.13: I–V characteristics for the array described by the solid line in Figure 4.12 for integer and half-integer multiples of the flux quantum in the loop.

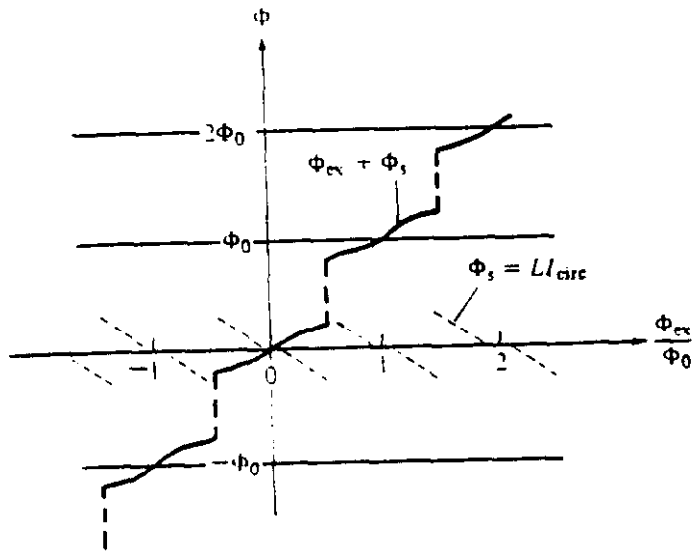

Figure 4.15: Dependance of total flux in the loop and the self-induced flux on the applied flux for the same array as in Figure 4.14 ($LI_c = 5\Phi_0/\pi$).

phase correlation due to the weak links) and the loop switches when one state requires less circulating current than the neighbouring state.

Appendix A

The probability current density

I would like to show a way to derive the probability continuity equation based on the invariance properties of the Lagrangian density (see for example [2, Chapter 11] for an introduction to classical field theory) for the Schrödinger equation. I haven't found this calculation in the books I have read, so I will go through the steps in a more detailed way.

Given the Lagrangian density for the Schrödinger equation in an electro-magnetic field [8, §10.12]:

$$\mathcal{L} = \frac{\hbar}{i}\psi^*\dot{\psi} + \frac{1}{2m}\left[\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\psi\right]^* \cdot \left[\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\psi\right] + q\phi\psi^*\psi, \tag{A.1}$$

where $\mathbf{A}$ is the vector potential and ϕ is the scalar potential, so that the fields are

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{A.2}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{A.3}$$

It is easily shown that (A.1) indeed leads to the Schrödinger equation [22, pages 499–500] in an electro-magnetic field (1.7), using Euler-Lagrange equations for ψ or ψ^* (thought as independent fields). The variation of $\mathcal{L}$ is:

$$\delta\mathcal{L} = \sum_{\psi,\psi^*}\left[\frac{\partial\mathcal{L}}{\partial\psi_j}\delta\psi_j + \sum_{i=1}^{3}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi_j}{\partial x_i})}\delta\left(\frac{\partial\psi_j}{\partial x_i}\right) + \frac{\partial\mathcal{L}}{\partial\dot{\psi}_j}\delta\dot{\psi}_j\right], \tag{A.4}$$

and finally, using the rule for the differentiation of a product of functions, one finds[1]:

$$\delta\mathcal{L} = \sum_{\psi,\psi^*}\left[\frac{\partial\mathcal{L}}{\partial\psi_j} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi_j}{\partial x_i})}\right) - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_j}\right]\delta\psi_j +$$

[1] From now on the summation over the index i will be implied, $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

$$+\frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_j}{\partial x_i})}\delta\psi_j\right)+\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_j}\delta\psi_j\right). \tag{A.5}$$

The term in square brackets is always equal to zero because it is just the sum of the two Euler-Lagrange equations for ψ and ψ^*, so if there exists a transformation of the fields for which $\mathcal{L}$ is invariant, then a "constant of the motion" is:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi}{\partial x_i})}\delta\psi+\frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi^*}{\partial x_i})}\delta\psi^*\right)+\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\delta\psi+\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*}\delta\psi^*\right)=0. \tag{A.6}$$

By inspection, it is seen that $\mathcal{L}$ is invariant under the global transformations:

$$\begin{cases} \psi & \rightarrow & e^{i\chi}\psi \\ \psi^* & \rightarrow & e^{-i\chi}\psi^*. \end{cases} \tag{A.7}$$

So for an infinitesimal transformation ($e^{i\chi} \approx (1+i\chi)$) we find that:

$$\begin{cases} \delta\psi & = & i\chi\psi \\ \delta\psi^* & = & -i\chi\psi^* \end{cases} \tag{A.8}$$

and substituting the (A.8) in the (A.7) we get:

$$\hbar\chi\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{2m}\left(\psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}-qA_i\right)\psi+\psi\left(-\frac{\hbar}{i}\frac{\partial}{\partial x_i}-qA_i\right)\psi^*\right)\right]+\frac{\partial}{\partial t}(\psi^*\psi)\right\}=0, \tag{A.9}$$

or equivalently (supposing $\chi \neq 0$),

$$\nabla\cdot\left\{\frac{1}{2m}\left[\psi^*\left(\frac{\hbar}{i}\nabla-q\mathbf{A}\right)\psi+\psi\left(-\frac{\hbar}{i}\nabla-q\mathbf{A}\right)\psi^*\right]\right\}+\frac{\partial}{\partial t}(\psi^*\psi)=0. \tag{A.10}$$

We have thus found the continuity equation for the probability density current $\mathbf{J}$ in the electro-magnetic field by means of Noether's theorem; the analytical expression of $\mathbf{J}$ is:

$$\mathbf{J}=\frac{1}{2m}\left[\psi^*\left(\frac{\hbar}{i}\nabla-q\mathbf{A}\right)\psi+\psi\left(-\frac{\hbar}{i}\nabla-q\mathbf{A}\right)\psi^*\right]. \tag{A.11}$$

I would finally like to remember that, despite the similarities between the probability current and the electric current density, these two quantities are not related in any way, in the one particle case. It will turn out that they are very strongly connected if we consider a system of a huge number of particles, all described by the same wave function.

Appendix B

Derivation of the relation between $\mathbf{J}_S$ and θ

In this appendix I will do the calculations that lead to equation (2.5) explicitly, however these are not a necessary condition to understand the rest of the paper. I have decided to write these calculations here only for completeness and because they are left to the reader in [3, chapter 21] too.

Let's consider the expression for the electric current density and write it in a slightly different way from (2.4):

$$\mathbf{J}_s = \frac{1}{2m}\left[\tilde{\psi}^\star\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\tilde{\psi} + \tilde{\psi}\left(\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\tilde{\psi}\right)^\star\right], \tag{B.1}$$

it is now clear that (B.1) is indeed the average between a complex expression and its complex conjugate, so $\mathbf{J}$ is a real quantity. This means that we will not have to bother about the imaginary part in the calculation.

Let's substitute:

$$\tilde{\psi}(\mathbf{r},t) \stackrel{\text{def}}{=} \sqrt{\rho(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}, \tag{B.2}$$

in the (B.1) and develop the obtained expressions:

$$\begin{aligned}
\mathbf{J}_s &= \frac{1}{2m}\left[\sqrt{\rho}e^{-i\theta}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\sqrt{\rho}e^{i\theta}\right) + \sqrt{\rho}e^{i\theta}\left(\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\sqrt{\rho}e^{i\theta}\right)\right)^\star\right] \\
&= \frac{1}{m}\Re\left\{\sqrt{\rho}e^{-i\theta}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\sqrt{\rho}e^{i\theta}\right)\right\} \\
&= \frac{\sqrt{\rho}}{m}\Re\left\{\frac{\hbar}{i}\left(i\sqrt{\rho}\nabla\theta + \nabla\left(\sqrt{\rho}\right)\right) - q\mathbf{A}\sqrt{\rho}\right\} \\
&= \frac{\rho}{m}\left(\hbar\nabla\theta - q\mathbf{A}\right),
\end{aligned}$$

and so, finally, we find:

$$\mathbf{J_s} = \frac{\hbar}{m} (\nabla\theta - q\mathbf{A})\,\rho, \tag{B.3}$$

and this is just the expression we have used in Chapter 2. This expression is independent from Maxwell equations, so there must be some new physics, not contained in classical electro-magnetism hidden here. We will see that this is just the case, in fact equation (B.3) leads to flux quantization and the Meissner effect in superconductors, however it is not yet obvious why (B.3) should describe the behaviour of a superconductor in a magnetic field.

Appendix C

The BCS theory of the energy gap

We will first consider a mathematical problem that will give us the tools to cope with the mechanism that builds the energy gap between the ground-state and the excited states [4, pages 624–627].

Consider an unperturbed one-particle system with an energy level spectrum such that one of the levels is R-fold degenerate and well separated in energy from the others, so that all these R states correspond to the same energy level. Introduce now a weak perturbation, that might split up the R states so that they now occupy a certain range of energies. We denote the R states associated with the unperturbed degenerate level $\varphi_1, \varphi_2, \ldots, \varphi_R$. These states satisfy the Schrödinger equation

$$\hat{\mathcal{H}}_0 \varphi_i = E\varphi_i, \tag{C.1}$$

we may choose $E = 0$ so that the equation for the unperturbed system becomes $\hat{\mathcal{H}}_0 \varphi_i = 0$.

In the first approximation the new states of the system may be written (following degenerate perturbation theory) as a linear combination of the unperturbed states

$$\psi_j = \sum_{i=1}^{R} c_{ji} \varphi_i. \tag{C.2}$$

To solve this problem, suppose that the ψ_j thus formed are exact solutions of the perturbed Schrödinger equation

$$(\hat{\mathcal{H}}_0 + \hat{U})\psi_j = \hat{U}\psi_j = \epsilon_j \psi_j, \tag{C.3}$$

which can be written as

$$\sum_i c_{ji} \hat{U} \varphi_i = \epsilon_j \sum_i c_{ji} \varphi_i. \tag{C.4}$$

Multiplying both sides of the (C.5) by φ_m^* and integrating over the volume we find

$$\sum_i c_{ji}\langle m|\hat{U}|i\rangle = \epsilon_j \sum_i c_{ji}\delta_{mi} = \epsilon_j c_{jm}. \tag{C.5}$$

So we find a set of R simultaneous equations for each j, in the R unknowns c_{jm}. These equations have a nontrivial solution only if the determinant of the coefficients vanishes, so (dropping the index j) we have:

$$\begin{vmatrix} \langle 1|\hat{U}|1\rangle - \epsilon & \cdots & \langle 1|\hat{U}|R\rangle \\ \vdots & \ddots & \vdots \\ \langle R|\hat{U}|1\rangle & \cdots & \langle R|\hat{U}|R\rangle - \epsilon \end{vmatrix} = 0. \tag{C.6}$$

The problem is to find the roots ϵ of this determinant, which in general is a hard numerical business. Suppose that

$$\langle m|\hat{U}|i\rangle = 1 \quad \forall m, i = 1, \ldots, R. \tag{C.7}$$

We now from determinant theory that:

$$\sum_j \epsilon_j = \sum_i \langle i|\hat{U}|i\rangle, \tag{C.8}$$

and

$$\sum_j \epsilon_j^2 = \sum_{ij} \langle i|\hat{U}|j\rangle, \tag{C.9}$$

so in our case we find:

$$\sum_j \epsilon_j = R, \tag{C.10}$$

$$\sum_j \epsilon_j^2 = R^2. \tag{C.11}$$

Let's now form the linear combination

$$\psi_1 = \frac{1}{\sqrt{R}} \sum_i \varphi_i, \tag{C.12}$$

this is indeed a solution of the problem, as can be seen substituting the (C.12) in the (C.5)

$$\sum_i \frac{1}{\sqrt{R}} = \sqrt{R} = \epsilon_1 \frac{1}{\sqrt{R}}, \tag{C.13}$$

so we finally find for the energy

$$\epsilon_1 = R. \tag{C.14}$$

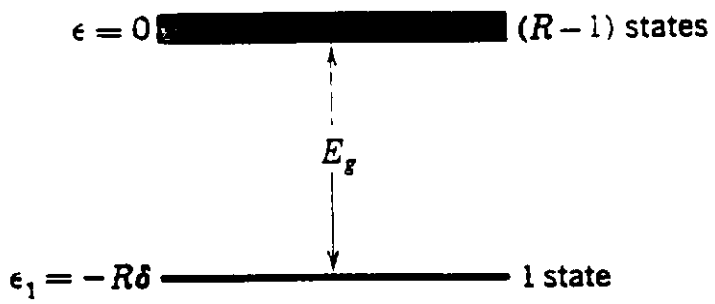

Figure C.1: Energy eigenvalue spectrum for the perturbed hamiltonian.

This means that the energy of the other states is $\epsilon_j = 0$ for every $j \neq 1$ (just because the sum of the roots of the determinant must be equal to R).

If the potential $\hat{U}$ is an attractive interaction, the matrix elements will be negative

$$\langle i|\hat{U}|j\rangle = -\delta, \tag{C.15}$$

so the energy spectrum will be (see Figure C.1):

$$\epsilon_j = \begin{cases} -R\delta & \text{if} j = 1 \\ 0 & \text{if} j \neq 1 \end{cases} \tag{C.16}$$

Thus we have found the existance of an energy gap in the spectrum of the perturbed levels. Even if the interaction δ may be very weak, $R\delta$ may be large if the degeneracy is high.

In the problem just treated, we have been dealing with one-particle states; suppose now that we have a system of $\mathcal{N}$ electrons, initially wihout mutual interactions. The various states Φ of the $\mathcal{N}$-particle wave function can be specified by giving the occupancy number (between zero and one, because of the Pauli Exclusion principle) of the one-electron states. We may label a one-electron state as $\mathbf{k}\uparrow$, for example, to mean that it has wavevector $\mathbf{k}$ and spin up. If we decide to label an $\mathcal{N}$-particle state in terms of *occupied* one-particle states, in the case of non-interacting electrons (the equivalent of the unperturbed hamiltonian) a state can only be either occupied or vacant, so in our case we have, for example

$$\Phi_s \equiv \mathbf{k_1}\uparrow; \mathbf{k_2}\uparrow; \mathbf{k_3}\downarrow; \ldots; \mathbf{k_N}\uparrow, \tag{C.17}$$

where the subscripts on the $\mathbf{k}$ denote specific values of the wavevector according to some arbitrary ordering.

We now let the electrons interact via a potential

$$\hat{U} = \sum_{nm} \hat{U}(\mathbf{r}_n - \mathbf{r}_m). \tag{C.18}$$

Each term of the sum acts to scatter two electrons into two empty electron-states, provided they are empty, i.e. one scattering event takes the system from Φ_s to Φ_u.

Can we make the assumption, as we did before, that the matrix elements of $\hat{U}$ are all equal ? No, this is not possible in general. First of all, we cannot scatter at all between states of different total spin because $\hat{U}$ does not contain spin operators. We can also see that some of the matrix elements are positive, while some others are negative, in fact suppose that the matrix element for Φ_s going into $\Phi_w \equiv \mathbf{k_a} \uparrow ; \mathbf{k_b} \uparrow ; \mathbf{k_c} \downarrow ; \ldots$ is positive, then the matrix element for Φ_s going into $\Phi_x \equiv \mathbf{k_b} \uparrow ; \mathbf{k_a} \uparrow ; \mathbf{k_c} \downarrow ; \ldots$ is negative because electrons are fermions, and so, according to the Pauli exclusion principle, the interchange of the order of two states in a wavefunction changes the sign of the wavefunction.

Actually there is a subspace of the Hilbert space of the $\mathcal{N}$-particle states for which all the matrix elements can be allowed to be equal. Suppose to consider only those states which are occupied by pairs of electron, e.g. such that if $\mathbf{k} \uparrow$ is occupied then also $-\mathbf{k} \downarrow$ is occupied[1], then the many-particle states that we consider are of the form

$$\Phi_A \equiv \mathbf{k_1} \uparrow ; -\mathbf{k_1} \downarrow ; \mathbf{k_2} \uparrow ; -\mathbf{k_2} \downarrow ; \ldots \tag{C.19}$$

These states are closed under the usual operations for Hilbert spaces (i.e. if we form linear combinations of paired states we still find paired states) so they indeed form a subspace of the problem.

Within this subspace, it is not unreasonable to assume a potential whose matrix elements are all equal, as we exchange *two* particles at a time, and so the many-body wavefunction does not change sign on a pair exchange (behaving just as if the system was made of *bosons*). Then by analogy with the earlier solution (C.16) we obtain a spectrum which has a single ground state separated by an energy gap from the excited states. Our discussion has neglected completely the kinetic terms, but the BCS theory shows that inclusion of the kinetic energy does not destroy the energy gap.

[1] We could form pairs in many other ways, for example with parallel spins, but the one we consider turns out to be the least energetic.

Appendix D

A general analysis of the Meissner effect

Suppose we have a simply connected and connected lump of superconducting material. Let's write again from equation (2.5):

$$\mathbf{J}_s = \frac{\hbar}{m^*}\left(\nabla\theta - \frac{q^*}{\hbar}\mathbf{A}\right)\rho, \tag{D.1}$$

suppose also to be in stationary conditions, so that the charge conservation reads $\nabla \cdot \mathbf{J}_s = 0$.

Let's now make a choice for the gauge very convenient in quasi-static situations in superconductivity, the London gauge [14, §3.05]:

$$\begin{cases} \nabla \cdot \mathbf{A} &= 0 \\ A_\perp &= -\Lambda J_{s\perp} \quad \text{at the surface.} \end{cases} \tag{D.2}$$

where $\Lambda = m^*/(q^*\rho)$. Let's show that there always exists a vector potential satisfying these conditions; suppose that $\mathbf{A}$ does not satisfy the (D.2), take another vector potential $\mathbf{A}'$ such that

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \tag{D.3}$$

where χ is any $\mathcal{C}^2$ scalar function (this is the ordinary gauge freedom of electromagnetism, see [17, §6.5]), and such that $\mathbf{A}'$ satisfies (D.2)

$$\begin{cases} \nabla \cdot \mathbf{A}' &= \nabla \cdot \mathbf{A} + \nabla^2\chi &= 0 \\ A_\perp{}' &= A_\perp + (\nabla\chi)\cdot\hat{\mathbf{n}} &= -\Lambda J_{s\perp} \quad \text{at the surface.} \end{cases} \tag{D.4}$$

that is:

$$\begin{cases} \nabla^2\chi &= -\nabla \cdot \mathbf{A} \\ (\nabla\chi)\cdot\hat{\mathbf{n}} &= -(\Lambda J_{s\perp} + A_\perp) \quad \text{at the surface.} \end{cases} \tag{D.5}$$

which is a Poisson equation with Neumann boundary conditions, so a solution exists and it is unique, apart from an arbitrary additive constant [17, §1.9].

Taking the divergence of (D.1) we find, in the London gauge:

$$\nabla^2\theta = 0, \tag{D.6}$$

and for the boundary condition we have

$$\mathbf{J}_s \cdot \hat{\mathbf{n}} = \frac{\hbar}{m^*}\left((\nabla\theta)\cdot\hat{\mathbf{n}} - \frac{q^*}{\hbar}\mathbf{A}\cdot\hat{\mathbf{n}}\right)\rho = \frac{\hbar}{m^*}\frac{\partial\theta}{\partial n} - \frac{1}{\Lambda}A_\perp, \tag{D.7}$$

that is:

$$\frac{\partial\theta}{\partial n} = 0. \tag{D.8}$$

and this is, obviously, valid at the surface of the superconductor. The solution for this boundary value problem is easily seen to be θ = const. over the whole volume (being connected) and it is unique (we are again dealing with a Laplace equation with Neumann boundary conditions) apart from an additive constant. So we see that equation (D.1) finally becomes:

$$\mathbf{J}_s = -\frac{q^*\rho}{m^*}\mathbf{A} = -\frac{1}{\Lambda}\mathbf{A}. \tag{D.9}$$

Using the vector potential $\mathbf{A}$ and the scalar potential ϕ, Maxwell equations can be written in the form:

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{D.10}$$

$$-\nabla^2\mathbf{A} + \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla(\nabla\cdot\mathbf{A}) + \mu_0\epsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right) = \mu_0\mathbf{J}. \tag{D.11}$$

imposing the London gauge and remembering that we are dealing with stationary conditions (i.e. the fields are constant in time), the second one becomes

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} = \frac{\mu_0}{\Lambda}\mathbf{A} = \lambda^2\mathbf{A}. \tag{D.12}$$

It can be shown, using only the structure of the Helmoltz equation (D.12) and some theorems from potential theory that the maxima of $\mathbf{A}$ cannot lie inside the superconductor and that the field must be exponentially small for all points that lie far distant from the surface, compared with the penetration depth [23, §7].

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